THE UNIVERSITY OF ALBERTA

RELEASE FORM

NAME OF AUTHOR .... SUBHASHCHANDRA MORESHWAR KARNIK ....

TITLE OF THESIS .... "COHERENCE TOPOLOGIES" ....

....

....

DEGREE FOR WHICH THESIS WAS PRESENTED .... Ph.D. ....

YEAR THIS DEGREE GRANTED .... 1974 ....

THE UNIVERSITY OF ALBERTA

COHERENCE TOPOLOGIES

by

© SUBHASHCHANDRA MORESHWAR KARNIK

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE

OF DOCTOR OF PHILOSOPHY

IN

MATHEMATICS

DEPARTMENT OF MATHEMATICS

EDMONTON, ALBERTA

FALL , 1974

THE UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled COHERENCE TOPOLOGIES submitted by SUBHASHCHANDRA MORESHWAR KARNIK in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics.

Dedicated to my parents MORESHWAR and SUMATI

ABSTRACT

Conditions on the topology of a topological space X which require that it be in some sense coherent with the topologies on certain subspaces of X have recently received a great deal of attention. Perhaps the most familiar examples are the defining conditions for k spaces and sequential spaces, although less familiar examples abound. Our endeavour in this thesis has been to establish a general framework for the investigation of these coherence concepts and then to present several new results that will throw light on less investigated classes, for example, S_R and k_R spaces .

To be precise, we have introduced very general T' , T , T_R spaces by relating their respective topologies to subspaces belonging to a quite arbitrary prechosen class $\mathcal{T}$ of topological spaces on the same pattern as the topologies of k' , k , k_R spaces depend on (a very restrictive) class of compact subspaces .

Our primary results are structure theorems, covering mapping characterizations, and combinatorial and product theorems for T' , T and T_R spaces ; and these are obtained in as general a setting as possible so as to yield many interesting known results as corollaries .

Our new results usually concern T_R spaces . The centre of

our interest has always been the class of S_R spaces being a class of spaces of recent interest wider than the traditional class of sequential spaces . Some questions remain unsettled ; they are stated precisely at appropriate places .

ACKNOWLEDGEMENTS

Foremost thanks are due to my supervisor, Professor Stephen Willard, who suggested the topic of this thesis and guided me with understanding and patience. He offered me encouragement throughout this venture and helped me in every possible way till the end. I am very much grateful to him for all this.

I am also very much thankful to Professor Á. Császár of Budapest who made several important suggestions at various places in the thesis. These suggestions have improved my understanding of certain aspects of the subject matter.

I must mention two of my former teachers. They are Professor M. D. Mavinkurve and Professor S. A. Naimpally. Their influence on me, also, has been very significant. The former introduced me to the beauty of Topology, while the latter attracted me to the research aspect of it. I am equally indebted to them as well.

All these years I received the necessary monetary support from various sources the chief among them being the University of Alberta and personal N. R. C. grants of Professor S. A. Naimpally and Professor S. Willard. At different times as the need arose, I was also supported by Professor M. G. Murdeshwar, Professor A. Sharma and Professor E. F. Steiner from their personal N. R. C. grants. I would like to record my

appreciation for their timely help .

I also wish to thank my colleague and friend, Mrs. Mary Easwaran, for spending her precious time to type this thesis with concern and care .

Lastly, this thesis will be grossly incomplete if I do not acknowledge my wife, Medha, who has always been a fountain of inspiration. But for the courage she created in me I could not have brought this project to its current stage.

TABLE OF CONTENTS

LIST OF FIGURES

INTRODUCTION

This thesis began as an investigation of Fréchet, sequential and S_R spaces, our point of view broadening when it became apparent that most results in this area carry over, without essential change in the arguments, to similar situations (as represented, for example, by the k', k and k_R spaces). Thus we begin with a discussion of Fréchet, sequential and S_R spaces.

A space is Fréchet if the closure of each of its subsets is the set of limits of sequences contained in that subset. Fréchet spaces were introduced by Arhangel'skiǐ [3] and represent a class of spaces broader than the first countable spaces, whose topologies are "sequentially determined". Seizing on this point of view Franklin ([12] and [13]) introduced and studied the still broader class of sequential spaces. (A space is sequential if each sequentially closed set, that is, each set containing all limits of sequences taken from that set, is closed.)

Arhangel'skiǐ showed that the Fréchet spaces were precisely the pseudo-open images of metric spaces, and Franklin established that the sequential spaces were precisely the quotients of metric spaces.

But an even wider class of spaces exists whose topology is

in some sense determined by its convergent sequences . A space is an S_R space if each sequentially continuous function (one which preserves sequential limits) with Tychonoff range is continuous . (We may note in passing that if one replaces "Tychonoff" , in this definition, with "Hausdorff" , one has an alternate definition of sequential spaces .) Mazur [18] and Noble [27] have proved an important product theorem for S_R spaces (see Theorem I.3.2) , but it is significant that, until now, no characterization theorem for S_R spaces similar to the Arhangel'skiĭ - Franklin results on Fréchet and sequential spaces has been produced . Our Theorem II.4.3 fills this gap .

That theorem, as well as most of the results in this thesis, is cast in a very general setting . We adopted this point of view upon observing that, for example, sequential spaces are defined in the same way k spaces are defined, and what is more, the standard characterization theorems for these spaces (Franklin's in the case of sequential spaces, Cohen's [9] in the case of k spaces) are proved in exactly the same way . The quasi-k spaces of Nagata [25] and cluster spaces (we call them c spaces) of Schedler [28] are likewise structurally no different from sequential or k spaces and are consequently encompassed by our general scheme .

Apart from the fundamental structure theorems, certain relevant combinatorial and product questions are also treated. Occasionally, certain concepts had to be dealt with individually when doing so had a

definite advantage . Material which could have possibly obstructed the general flow of the presentation is reserved for the last chapter . This mainly consists of certain examples and some results on linearly ordered topological spaces, which have no direct bearing on the development of the general theory .

CHAPTER I

PRELIMINARIES

I.1 It is most appropriate to start with definitions of various concepts which we intend to look at in a general setting a little later. For the sake of neatness, we prefer to present them in groups.

Recall that a <u>filter base</u> $\mathcal{F}$ is a non-empty collection of non-empty subsets of a certain set Y such that for every two members F_1 and F_2 of $\mathcal{F}$ there is a member F_3 of $\mathcal{F}$ such that F_3 is contained in $F_1 \cap F_2$. In a topological space Y, a filter base $\mathcal{F}$ <u>accumulates at a point</u> y if $y \in \overline{F}$ for every $F \in \mathcal{F}$. A decreasing sequence of non-empty subsets - a special filter base - is called a <u>decreasing sequence</u> herein.

We are now ready for the following groups of definitions.

<u>Group I</u> : A space Y is <u>strongly Fréchet</u> iff whenever a decreasing sequence (F_n) accumulates at y in Y, there exists $y_n \in F_n$ for each n, such that $y_n \to y$.

A space Y is <u>strongly k'</u> (<u>strongly quasi-k'</u>, <u>strongly c'</u> respectively) iff whenever a decreasing sequence (A_n) accumulates at y in Y, there is a compact (countably compact, countable respectively) subset K of Y such $y \in \overline{(K \cap A_n)}$ for every n.

<u>Group II</u> : A space Y is <u>Fréchet</u> iff whenever $y \in \overline{A}$ in Y, there is a sequence in A which converges to y.

A space Y is <u>k'</u> (<u>quasi-k'</u>, <u>c'</u> respectively) iff whenever $y \in \overline{A}$ in Y, there is a compact (countably compact, countable respectively) subset K of Y such that $y \in (K \cap A)^{-}$.

<u>Group III</u> : A space Y is <u>sequential</u> iff a subset A of Y is closed whenever a sequence $(y_n) \subset A$ and $y_n \to y$, then $y \in A$.

A space Y is <u>k</u> (<u>quasi-k</u>, <u>c</u> respectively) iff a subset A of Y is closed whenever $A \cap K$ is closed in K for every compact (countably compact, countable respectively) subset K of Y.

<u>Group IV</u> : A space Y is <u>S_R</u> iff every sequentially continuous real-valued function on Y is continuous. (A function $f : Y \to R$ is sequentially continuous iff whenver $y_n \to y$, then $f(y_n) \to f(y)$.)

A space Y is <u>k_R</u> (<u>quasi-k_R</u>, <u>c_R</u> respectively) iff every real-valued function on Y which is continuous on every compact (countably compact, countable respectively) subset of Y is continuous.

We are now in a position to introduce the general scheme. The correspondence between the following set of definitions and Groups I through IV is indeed one-to-one and obvious. For example, Group I corresponds to Definition I.1.1, Group II to Definition I.1.2 and so on.

Let Y be a topological space and $\mathcal{T}$ a class of topological spaces which is closed under homeomorphisms. The statements " A is

a $\mathcal{T}$-space" , " A is a $\mathcal{T}$-subspace of Y " and " A is a $\mathcal{T}$-subset of Y " will be synonymous and will mean simply that $A \in \mathcal{T}$.

I.1.1 Definition A space Y is strongly $\underline{T}'$ iff whenever a decreasing sequence (A_n) accumulates at y in Y , there exists a $\mathcal{T}$-subset K of Y such that $y \in (K \cap A_n)^-$ for every n .

I.1.2 Definition A space Y is $\underline{T}'$ iff whenever $y \in \overline{A}$ for a subset A of Y , there is a $\mathcal{T}$-subset K of Y such that $y \in (K \cap A)^-$.

I.1.3 Definition A space Y is $\underline{T}$ iff whenever $F \cap K$ is closed in K for each $\mathcal{T}$-subspace K of Y , then F is closed in Y . (A subset F of Y with the property that $F \cap K$ is closed (open) in K for each $\mathcal{T}$-subspace K of Y will be called T-closed (T-open) .)

I.1.4 Definition A space Y is $\underline{T_R}$ iff every real-valued function on Y whose restriction to each $\mathcal{T}$-subspace of Y is continuous is continuous on Y . (A function which is continuous on each $\mathcal{T}$-subspace of a space Y will be said to be T-continuous on Y .) In view of the fact that every Tychonoff space can be embedded in a product of lines, the condition that functions be real-valued is superfluous here. That is, Y is a T_R space iff every T-continuous function on Y with arbitrary Tychonoff range is continuous. (If arbitrary Hausdorff ranges are allowed, the class of Hausdorff T_R spaces coincides with the class of Hausdorff T spaces. To see that a Hausdorff T_R space X is a T space, consider the identity mapping $id : X \to TX$ (see I.3.1(b)) .

I.2 For several different classes $\mathcal{T}$, the topological spaces in some sense coherently determined by $\mathcal{T}$ have familiar names as listed in Groups I through IV in I.1 . It seems most convenient to list them again in an 'implication' diagram as done below. While doing so, we have also taken an opportunity to introduce certain connected coherence topologies.

strongly T'	strongly c'	←	strongly Fréchet [30]	→	strongly k' [30]	→	strongly quasi-k'		strongly C'
↓	↕		↓		↓		↓		↕
T'	c' [28]	←	Fréchet [3]	→	k' [5]	→	quasi-k'		C'
↓	↕		↓		↓		↓		↕
T	c [28]	←	sequential [8]	→	k [1]	→	quasi-k [25]		C
↓	↓		↓		↓		↓		↕
T_R	c_R	←	S_R [27]	→	k_R [20]	→	quasi-k_R		C_R
$\mathcal{T}$	countable spaces		convergent sequences		compact spaces		countably compact spaces		connected spaces

Fig. 1

Where a reference is supplied in the diagram it serves to lead to the earliest mention of the concept to the best of author's knowledge, as well as to indicate that the terminology is not ours. In this connection we might mention that Schedler [28] refers to (what we call) c spaces as cluster spaces. We should also further add that by 'convergent sequence' we mean the <u>range</u> of a sequence together with its limit.

We have some comments about the use of separation axioms in this thesis. First, note that in general, in the absence of separation axioms, if T is the class of convergent sequences the concepts of strongly T' , T' , T and T_R spaces do not coincide with those of strongly Fréchet, Fréchet, sequential and S_R spaces respectively (in fact, the topologies of the former are weaker than the respective topologies of the latter) . However, in the class of Hausdorff spaces the distinction between the respective concepts in these two groups vanishes. Hence, whenever we speak of a strongly Fréchet, Fréchet, sequential or S_R space we will assume Hausdorff separation without explicit mention - there exists no such blanket assumption regarding separation axioms otherwise.

We must point out here the fact that first countable spaces are strongly Fréchet, locally compact (in the sense that every point of the space has a compact neighbourhood) spaces are strongly k' and that locally countably compact (in the sense that every point of the space has a countably compact neighbourhood) spaces are strongly quasi-k' . Also, by considering characteristic functions of components, one easily sees that X is C_R iff each component is open and closed iff X is a disjoint union of connected spaces. The equivalence of strongly C' and C_R spaces follows. (This equivalence with its proof is pointed out by Á. Császár.) These spaces coherently determined by connected spaces will be used mainly to illustrate certain points.

Lastly, in order to get quickly to the core of the matter, we have reserved comments such as reversibility of certain implication-arrows in the diagram for the last chapter. Most of the implications, however, follow quite easily. Also, in general there exists no relation

between connected coherence topologies and those other in the diagram.

I.3 There are certain results of basic importance which we mention here. They will be used without explicit reference.

Let $(X,\underset{\sim}{T})$ be a topological space and $\mathcal{T}$ a class of topological spaces. By $\underset{\sim}{T}_{\mathcal{T}}$ we mean the topology on X consisting of all sets which are T-open in $(X,\underset{\sim}{T})$. We often write "TX" for the topological space $(X,\underset{\sim}{T}_{\mathcal{T}})$, reserving "X" for the space $(X,\underset{\sim}{T})$.

I.3.1 <u>Proposition</u>

a) The identity map from TX to X is continuous.

b) If K is a $\mathcal{T}$-subset of X, the relativization of $\underset{\sim}{T}$ to K is identical with that of $\underset{\sim}{T}_{\mathcal{T}}$. Consequently, if $\mathcal{T}$ is closed under continuous bijections, a set is a $\mathcal{T}$-set in X iff it is a $\mathcal{T}$-set in TX.

c) TX is a T space.

d) A function on TX is continuous iff it is T-continuous on X.

e) $\underset{\sim}{T}_{\mathcal{T}}$ is the largest topology on X which agrees with $\underset{\sim}{T}$ on $\mathcal{T}$-subsets in X.

<u>Proofs</u> : a) This is easy to prove.

b) To verify that the relativization of $\underset{\sim}{T}_{\mathcal{T}}$ to K is the relativization of $\underset{\sim}{T}$ to K, let $G \subset K$ be T-open in K. Then there

is a T-open subset O of X such that $G = O \cap K$. But then by definition of T-open sets, $O \cap K$ is open in K for every $\mathcal{T}$-set K in X. The result follows.

The second statement is now easy to see. The condition that $\mathcal{T}$ is closed under continuous bijections is indeed essential here. For, if $\mathcal{T}$ is the class of discrete spaces, then TX is discrete for any X, whence each of the subspaces of TX is a $\mathcal{T}$-subset without being a $\mathcal{T}$-subset in X. That the said condition is essential as well as the foregoing example was pointed out by Á. Császár.

c) TX will be a T space iff every T-open subset of TX is open in TX. But a subset of X is T-open in TX iff it is T-open in X. (This needs (b).) Hence TX is a T space.

d) If a function on TX is continuous, it is also continuous with respect to $\underset{\sim}{T}_{\mathcal{T}}$ on every $\mathcal{T}$-subspace K of TX. But since TX and X induce the same topology on K, the function under consideration is T-continuous on X. Conversely, if $f : X \to Y$ be T-continuous on X, suppose that O is an open subset of Y. To prove that f is continuous with respect to $\underset{\sim}{T}_{\mathcal{T}}$ we must prove that $f^{-1}(O)$ is T-open in X. But this is obvious since $f^{-1}(O) \cap K$ is open in K for every $\mathcal{T}$-subset K of X.

e) This is easy to prove.

Little work has been done on T_R spaces in general. The main result in this area is the fundamental theorem of Mazur [18],

improved by Noble [27] , about S_R spaces.

I.3.2 Theorem (Mazur, Noble)

a) Every weakly inaccessible* cardinal is non-sequential.**

b) If X_a is non-indiscrete Hausdorff and first countable for every $a \in A$, then $\prod_{a \in A} X_a$ is an S_R space if the cardinal of A is non-sequential.

The following results will also be needed in the sequel.

I.3.3 Theorem (Schedler [28]) X is a c' space iff X is a c space.

I.3.4 Theorem (Arhangel'skiĭ [6]) A topological space X is Fréchet iff every subspace of X is a k space.

---

* A cardinal $\aleph_\alpha$ is said to be weakly inaccessible iff $\alpha > 0$ is a limit ordinal and $\sum_{s \in S} m_s < \aleph_\alpha$ whenever $\overline{\overline{S}} < \aleph_\alpha$ and each $m_s < \aleph_\alpha$.

** A cardinal $\overline{\overline{A}}$ is said to be non-sequential iff there does not exist a non-zero real-valued sequentially continuous function $\sigma : 2^A \to R$ which maps finite sets to zero.

I.4 We will define here several classes of mappings which will be used in the Structure Theorems of Chapter II and in the $\mathcal{T}$-covering characterizations of Chapter III . We will also introduce a new class of mappings wider than the class of quotient mappings which we call T-weak-quotient mappings. These will be mentioned when their need arises. (Quotient mappings are not defined due to their familiarity.)

I.4.1 <u>Definition</u> A mapping f from X onto Y is called <u>countably bi-quotient</u> mapping if it satisfies either of the following equivalent conditions :

a) Whenever $y \in Y$ and (U_n) is an increasing countable cover of $f^{-1}(y)$ by open subsets of X , then $y \in \text{Int. } f(U_n)$ for some n .

b) Whenever (A_n) is a decreasing sequence accumulating at y in Y , then $(f^{-1}(A_n))$ accumulates at some $x \in f^{-1}(y)$.

(These mappings were introduced by A. H. Stone in [31] and the equivalence of a) and b) is proved by F. Siwiec in [30] .)

I.4.2 <u>Definition</u> A mapping f from X onto Y is called <u>hereditarily quotient</u> mapping if it satisfies any one of the following equivalent conditions :

a) $f \mid f^{-1}(S) : f^{-1}(S) \to S$ is quotient mapping for every $S \subset Y$.

b) Whenever U is a neighbourhood of $f^{-1}(y)$ in X, $f(U)$ is a neighbourhood of y in Y. (Neighbourhoods need not be open.)

c) Whenever $y \in \overline{A}$ in Y, then $x \in (f^{-1}(A))^{-}$ for some $x \in f^{-1}(y)$.

(The equivalence of a) and b) was proved by A. V. Arhangel'skiĭ in [3] who also introduced these mappings. The equivalence of b) and c) is due to E. Michael [22]. The concept as at b) above is usually called <u>pseudo-open</u>.)

Obviously, every countably bi-quotient mapping is hereditarily quotient mapping and every hereditarily quotient mapping is quotient mapping.

I.5 We will say that a space X is <u>locally strongly T'</u> (<u>locally T'</u>, <u>locally T</u>, <u>locally T_R</u> respectively) iff each point of X has a neighbourhood whose closure is strongly T' (T', T, T_R respectively) space. We will say that a space X is <u>locally $\mathcal{T}$</u> iff each point of X has a neighbourhood which as a subspace of X belongs to $\mathcal{T}$.

I.5.1 <u>Theorem</u> If X is locally strongly T' (locally T', locally T, locally T_R respectively), then X is strongly T' (T', T, T_R respectively) space. Also, if X is locally $\mathcal{T}$, X is strongly T'.

Proof : Suppose X is locally strongly T'. Let (A_n) be a decreasing sequence accumulating at a point x of X. Let U be a neighbourhood of x such that $\overline{U}$ is a strongly T' space. But then $x \in (\overline{U} \cap A_n)^-$ for every n whence one can find a $\mathcal{T}$-subset K of $\overline{U}$ such that $x \in (\overline{U} \cap A_n \cap K)^-$ for every n which proves that X is strongly T'.

Now if X is locally T', let $F \subset X$ and let $x \in \overline{F}$. Let U be a closed neighbourhood of x which is a T' space. Then $x \in Cl_U(F \cap U)$ and hence there is a subspace K of U which belongs to $\mathcal{T}$ such that $x \in Cl_U(F \cap U \cap K)$. Then certainly $x \in Cl_X(F \cap K)$. Thus X is a T' space.

For T spaces, the result has been observed by Mrowka [24]. (See his Theorem 1.3.)

For T_R spaces the result follows easily from the fact that a "locally continuous" function is continuous.

The second part of the statement of the result is easy.

Corollary The disjoint union of strongly T' (T', T, T_R respectively) spaces is a strongly T' (T', T, T_R respectively) space.

I.5.2 Theorem

a) If $\mathcal{T}$ is closed under countably bi-quotient mappings, then every countably bi-quotient image of a strongly T' space is

strongly T' .

b) If $\mathcal{T}$ is closed under hereditarily quotient mappings, then every hereditarily quotient image of a T' space is a T' space.

c) If $\mathcal{T}$ is closed under quotient mappings, then

i) every quotient of a T space is a T space

and ii) every quotient of a T_R space is a T_R space.

<u>Proofs</u> : a) Let f be a countably bi-quotient mapping from a strongly T' space X onto Y . Let y be a point in Y at which a decreasing sequence (A_n) in Y accumulates. Then there exists a point $x \in f^{-1}(y)$ such that $x \in (f^{-1}(A_n))^-$ for each n . But since X is strongly T' , there exists a $\mathcal{T}$-subset K in X such that $x \in (K \cap f^{-1}(A_n))^-$ for every n . Then $f(K)$ is a $\mathcal{T}$-subset in Y and $y = f(x) \in (f(K) \cap A_n)^-$ for every n .

b) Let X be a T' space and suppose q is a hereditarily quotient mapping of X onto Y . Let $F \subset Y$, $y \in \overline{F}$. Let x be a point of $q^{-1}(y) \cap (q^{-1}(F))^-$. Then for some subspace K of X which belongs to $\mathcal{T}$, $x \in (q^{-1}(F) \cap K)^-$. But then $y \in (q(q^{-1}(F) \cap K))^-$ which is a subset of $(F \cap q(K))^-$. Since $q(K) \in \mathcal{T}$, we are done.

c) (i) This has been proved by Mrówka ([24] , proposition 1.8) .

c) (ii) Let q be a quotient mapping of a T_R space X onto Y. If $f : Y \to R$ is T-continuous, then $f \circ q$ is T-continuous and hence continuous. Then f must be continuous, so Y is a T_R space.

The following comments are appropriate at this place :

For Fréchet spaces, Franklin ([12] , proposition 2.3) has proved the following converse to I.5.2(b) : if X, Y are Fréchet spaces and $q : X \to Y$ is a quotient mapping, then q is hereditarily quotient mapping. This result cannot be proved for T' spaces in general, for every c space is a c' space (I.3.3) and according to I.5.2(c)(i) this would imply every quotient mapping of a countable space is necessarily a hereditarily quotient mapping. This is seen to be false in the example below.

I.5.3 <u>Example</u> Let X_1 be the space of rationals in $(0,1)$ and X_2 be the space $\{0,1/2,1/3, \ldots\}$ with usual topology. Let X be the disjoint union of X_1 and X_2. Let Y be the space obtained by identifying each $1/n \in X_1$ with $1/n \in X_2$. (This is a modification of [12] , Example 1.8) . Let $f : X \to Y$ be the quotient mapping . Then f is not hereditarily quotient mapping, for X_2 is a neighbourhood of $f^{-1}(f(0))$, but $f(X_2)$ is not a neighbourhood of $f(0)$.

CHAPTER II

STRUCTURE THEOREMS

There are available now several theorems of the form taken by Cohen's theorem on k spaces ([9]) : ' X is a k space iff X is a quotient of a locally compact space'. Thus

a) Fréchet spaces are hereditarily quotient images of metric spaces ([12]) .

b) Sequential spaces are quotients of metric spaces ([12]).

c) c spaces are quotients of disjoint unions of countable spaces ([28]).

Two of the following theorems exhibit the above as special cases of certain general structure theorems for T' spaces and T spaces. This is followed by the development of a structure theorem for T_R spaces which is, so far as we know, new for any choice of $\mathcal{T}$. We have also a structure theorem for strongly T' spaces.

II.1 <u>Theorem</u> Let $\mathcal{T}$ be a class of spaces which is closed under countably bi-quotient mappings and includes $K \cup \{x\}$ whenever $K \in \mathcal{T}$, and $x \in \overline{K}$ in the topological space $K \cup \{x\}$. Then the following

are equivalent for any topological space Y :

a) Y is a strongly T' space.

b) Y is the countably bi-quotient image of a disjoint union of spaces from $\mathcal{T}$.

c) Y is the countably bi-quotient image of a locally $\mathcal{T}$ space.

<u>Proof</u> : a) → b) : Let X be the disjoint union of all subspaces of Y which belong to $\mathcal{T}$ and let $f : X \to Y$ be the natural mapping. Now if (A_n) is a decreasing sequence accumulating at a point y in Y , then since Y is strongly T' , there is a $\mathcal{T}$-subset $K \subset Y$ such that $y \in (K \cap A_n)^-$ for every n . Considering the summand $K \cup \{y\}$ of X , it follows that f is countably bi-quotient mapping.

b) → c) : Obvious.

c) → a) : Follows from I.5.1 and I.5.2(a) .

II.2 <u>Theorem</u> Let $\mathcal{T}$ be a class of spaces which is closed under hereditarily quotient mappings and includes $K \cup \{x\}$ whenever $K \in \mathcal{T}$, and $x \in \overline{K}$ in the topological space $K \cup \{x\}$. Then the following are equivalent for any topological space Y :

a) Y is a T' space.

b) Y is the hereditarily quotient image of a disjoint

union of spaces from $\mathcal{T}$.

c) Y is the hereditarily quotient image of a locally $\mathcal{T}$ space.

Proof : a) → b) : Let X be the disjoint union of the subspaces of Y which belong to $\mathcal{T}$ and let $f : X \to Y$ be the natural mapping. To see that f is hereditarily quotient mapping, let $y \in \overline{A}$ in Y . Then for some $K \in \mathcal{T}$ in Y , $y \in (A \cap K)^{-}$. Then $y \in \overline{K}$ and hence $K \cup \{y\} \in \mathcal{T}$. But then $K \cup \{y\}$ is a summand in the disjoint union X . It is easy to see that this summand contains a point common to $f^{-1}(y)$ and $(f^{-1}(A))^{-}$.

b) → c) : Obvious.

c) → a) : A locally $\mathcal{T}$ space is a T' space and the hereditarily quotient image of T' space is, by I.5.2 , a T' space.

II.3 Theorem Let $\mathcal{T}$ be a class of spaces which is closed under quotient mappings. Then the following are equivalent for any topological space Y :

a) Y is a T space.

b) Y is the quotient of a disjoint union of spaces from $\mathcal{T}$.

c) Y is the quotient of a locally $\mathcal{T}$ space.

Proof : a) → b) : Let X be the disjoint union of the subspaces

of Y which belong to $\mathcal{T}$, and $f : X \to Y$ the natural mapping. It is routine to verify that f is a quotient mapping.

b) $\to$ c) : Obvious.

c) $\to$ a) : A locally $\mathcal{T}$ space is a T space, and the quotient of a T space is a T space by I.5.2(c)(i) .

II.4 In order to characterize the T_R space in the spirit of structure theorems just given, we need some concepts which were introduced by McArthur [19] . A linear order $<$ on a set S is said to be <u>dense</u> provided whenever $x < y$ in S, then $x < z < y$ for some $z \in S$. We write $A \subset\subset B$ in a topological space X provided $\overline{A} \subset B$. A set $F \subset X$ is said to be a <u>strong G_δ-set</u> in X provided F is closed and $F = \bigcap_{i=1}^{\infty} G_i$ where each G_i is open and $\subset\subset$ is a dense linear order on $\{G_i\}$.

We introduce now some terms based on the above. A subset F of a topological space X will be called a <u>T-g-set</u> if $F = \bigcap_{i=1}^{\infty} G_i$ where each G_i is T-open and $\subset\subset_T$ is a dense linear order on $\{G_i\}$ (where $A \subset\subset_T B$ iff $Cl_{TX} A \subset B$) . A T-closed T-g-set will be called a <u>T-G-set</u> . Clearly, a T-G-set in X is precisely a strong G_δ-set in TX . Finally, we define a class of mappings wider than the class of quotient mappings. We call a continuous mapping f of X onto Y a <u>T-weak-quotient</u> mapping provided every T-g-set A is closed in Y whenever $f^{-1}(A)$ is closed in X . Our characterization of T_R spaces will use T-weak-quotient mappings, but to prove it efficiently,

we first prove the easy lemma:

II.4.1 <u>Lemma</u> A space X is a T_R space if and only if every T-G-set in X is closed.

<u>Proof</u> : According to McArthur [19] , a set F in X is a T-G-set if and only if F is a zero-set in TX . Thus it suffices to show X is a T_R space if and only if zero-sets in TX are closed in X .

To prove sufficiency, let $f : X \to R$ be T-continuous, and Z a zero-set in R . Then $f^{-1}(Z)$ is a zero-set in TX and hence closed in X . It follows easily that the inverse image of any closed set in R is closed in X , so f is continuous.

To prove necessity, let Z be a zero-set in TX . Then $Z = f^{-1}(0)$ for some real-valued T-continuous function f on X . But f is then continuous, so Z is closed.

II.4.2 <u>Lemma</u> Let $\mathcal{T}$ be a class of spaces which is closed under T-weak quotient mappings.

Then a T-weak quotient of a T_R space is a T_R space.

<u>Proof</u> : Let $f : W \to Z$ be a T-weak quotient mapping from a T_R space W onto Z . Let A be a T-G-set in Z . We must prove that A is closed. For this, it suffices to prove that $f^{-1}(A)$ is closed in W . But, since W is a T_R space, one would just show

that $f^{-1}(A)$ is a T-G-set in W. We, however, prove that if G be a T-open set in Z, $f^{-1}(G)$ is T-open in W, for other details would then follow easily. Hence, consider $f^{-1}(G)$. Let K be any subset of W which belongs to $\mathcal{T}$. Then $f(K) \in \mathcal{T}$ whence $f(K) \cap G$ is open in $f(K)$. It follows that $f^{-1}(G) \cap K$ is open in K which implies $f^{-1}(G)$ is T-open in W. This proves the lemma.

II.4.3 <u>Theorem</u> Let $\mathcal{T}$ be a class of spaces which is closed under T-weak quotient mappings. Then the following are equivalent for any topological space Y:

a) Y is a T_R space.

b) Y is the T-weak quotient of a disjoint union of spaces belonging to $\mathcal{T}$.

c) Y is the T-weak quotient of a locally $\mathcal{T}$ space.

<u>Proof</u> : a) $\rightarrow$ b) : Let X be the disjoint union of all subspaces of Y which belong to $\mathcal{T}$ and let f be the natural mapping of X onto Y. Then f is a T-weak quotient mapping.

To see this, let A be a T-g-set in Y such that $f^{-1}(A)$ is closed in X. However, since $f : X \rightarrow TY$ is a quotient mapping, A is T-closed. Thus A is a T-G-set and hence closed by II.4.1.

b) $\rightarrow$ c) : Obvious.

c) $\rightarrow$ a) : This follows from the fact that a locally $\mathcal{T}$ space is T_R and that a T-weak quotient of a T_R space is a T_R space by II.4.2.

As special cases of the foregoing theorems one gets structure theorems for all the various coherence topologies mentioned in the implication diagram on page 7.

CHAPTER III

$\mathcal{T}$-COVERING MAPPINGS AND COHERENCE TOPOLOGIES

Our main interest here centres around characterizing some of the coherence topologies in terms of $\mathcal{T}$-covering mappings which generalize the existing 'compact covering mappings' and 'sequence covering mappings'.

III.1 In this section we discuss various covering mappings. First we define $\mathcal{T}$-covering mappings.

III.1.1 Definition A continuous function f from X onto Y is called a *$\mathcal{T}$-covering mapping* iff to every $\mathcal{T}$-subset A of Y there corresponds a $\mathcal{T}$-subset B of X such that $f(B) = A$.

When $\mathcal{T}$ consists of compact spaces, a $\mathcal{T}$-covering mapping is compact covering mapping of Whyburn [35] and Arhangel'skiĭ [3]. Likewise, one gets countably compact-covering mappings when $\mathcal{T}$ is the class of countably compact spaces. When $\mathcal{T}$ is the class of convergent sequences, $\mathcal{T}$-covering mappings will be called *$\mathcal{S}$-covering mappings*. One might recall at this stage how Siwiec [30] has defined *sequence covering mappings*. A continuous function from X onto Y is called a sequence covering iff whenever $y_n \to y$ in Y, for some $x_n \in f^{-1}(y_n)$ and $x \in f^{-1}(y)$, $x_n \to x$. The following example shows that $\mathcal{S}$-covering mappings are not quite the same as sequence covering mappings.

III.1.2 Example Let X be an uncountable set. Let $\mathcal{C}$ be the co-finite topology on X while $\mathcal{T}$ be the topology on X having for its subbasis the family $\{a\} \cup \mathcal{C}$ where a is a fixed point of X. Then the identity mapping $id : (X,\mathcal{T}) \to (X,\mathcal{C})$ is certainly continuous.

In fact, id is an $\mathcal{S}$-covering mapping. For, if S_C be any sequence (x_n) together with a limit, say x in $(X,\mathcal{C})$, consider two cases: (i) S_C is an infinite set

(ii) S_C is a finite set.

Case (i) : Here there are infinite number of points of S_C which are different from a (if at all $a \in S_C$). Fix some point of S_C other than a and call it y. Let $y_1 = a$ (if at all $a \in S_C$) and let $y_2, \ldots ,y_n, \ldots$ be an enumeration of $S_C - \{a , y\}$. Then $y_n \to y$ in S_T where S_T has the same set as S_C with the topology inherited from $(X,\mathcal{T})$. Then S_T is a sequence with a limit in $(X,\mathcal{T})$ such that $id(S_T) = S_C$.

Case (ii) : Here S_C may be regarded as an eventually stationary sequence with a limit in both the topologies simultaneously.

However, id is not a sequence covering mapping. (For, if (x_n) be a sequence of distinct points such that $x_n \xrightarrow{\mathcal{C}} a$, $x_n \overset{\mathcal{T}}{\not\to} a$.
(Note that the spaces in this example are non-Hausdorff.)

The following proposition, however, shows that in the class of Hausdorff spaces the distinction between $\mathcal{S}$-covering mappings and sequence covering mappings vanishes.

III.1.3 <u>Proposition</u> If Y is a Hausdorff space, then $f : X \to Y$ is a sequence covering mapping if and only if f is an S-covering mapping.

<u>Proof</u> : <u>Only if</u> : Easy.

<u>If</u> : Let $f : X \to Y$ be an S-covering mapping. Let $y_n \to y$ in Y and let further $S_Y = (y_n) \cup (y)$.

We consider case (i) : S_Y is a finite set

and case (ii) : S_Y is an infinite set.

<u>Case</u> (i) : Here it is easy to find a sequence (z_n) in X such that $z_n \to z$ in X where $z_n \in f^{-1}(y_n)$ and $z \in f^{-1}(y)$.

<u>Case</u> (ii) : Here we first note that since f is an S-covering mapping there is $S_X = (x_n) \cup (x)$ such that $x_n \to x$ and $f(S_X) = S_Y$.

Then $f(x) = y$. For, if not, by continuity of f , $f(x_n) \to f(x)$ whence if $f(x) \neq y$, the sequence $(f(x_n))$ is eventually equal to $f(x)$ as S_Y has discrete topology on $S_Y - \{y\}$. However, this means that S_Y is a finite set giving a contradiction.

Now choose a point from $f^{-1}(y_n) \subset S_X$ and call it x'_n. It suffices to show that $x'_n \to x$. Let hence O be an open set containing x . Then, as $x_n \to x$, $x_n \in O$ whenever $n > m$ for some m . But then if $f\{x_1, \ldots , x_m\} = \{y_{k_1} , \ldots , y_{k_m}\}$, one has $x'_n \in O$ whenever $n > \max \{k_1 , \ldots , k_m\}$ whence $x'_n \to x$.

We now introduce some more concepts.

III.1.4 Definitions A continuous function f from X onto Y is said to be a cluster covering mapping iff whenever $y_n \leadsto y$ in Y, for some $x_n \in f^{-1}(y_n)$ and $x \in f^{-1}(y)$, $x_n \leadsto x$ in X where '$\leadsto$' is to be read ' clusters to ' . A $\mathcal{C}$-covering mapping will be $\mathcal{T}$-covering mapping where $\mathcal{T}$ is the class of all clustering sequences. (By a clustering sequence we will mean the range of a sequence together with one of its cluster points.) A countable covering mapping is obtained by interpreting $\mathcal{T}$ as the class of all countable spaces.

It is easy to see that every cluster covering mapping is a $\mathcal{C}$-covering mapping and that every $\mathcal{C}$-covering mapping is a countable covering mapping. In fact, every continuous surjection is a countable covering mapping and also a $\mathcal{C}$-covering mapping. This can be easily seen. Indeed, we have mentioned these mappings to attain a certain completeness in presentation. To see that a $\mathcal{C}$-covering mapping need not be a cluster covering mapping consider the identity mapping id from $(R,\mathcal{C})$ onto $(R,\mathcal{U})$ where $\mathcal{U}$ is the usual topology of the real line R and $\mathcal{C}$ is the usual topology on the real line R with zero discretized.

Also, just note in passing that one gets the same T and T' topologies, that is, c $(=c')$ topology, irrespective of whether one uses the class of countable spaces or the class of clustering sequences for $\mathcal{T}$.

III.2 We are now in a position to turn to the characterization of certain coherence topologies in terms of $\mathcal{T}$-covering mappings. We will assume that

(i) $\mathcal{T}$ <u>is closed hereditary</u>

(ii) $\mathcal{T}$ <u>is closed under continuous mappings</u>

and (iii) <u>all $\mathcal{T}$-subsets of the range space Y under consideration are closed</u> .

The discussion of cluster covering mappings and cluster coherence topologies is reserved for the end of this chapter. This is because even in very good spaces cluster sets or countable sets are not always closed and, as we will discover later, a c space in which countable sets are closed reduces to a discrete space. Lastly, we must mention that we could not obtain $\mathcal{T}$-covering mapping characterization of strongly T' spaces.

III.2.1 <u>Theorem</u> A topological space Y is a T' space if and only if every $\mathcal{T}$-covering mapping onto Y is hereditarily quotient mapping.

<u>Proof</u> : <u>If</u> : Let Y be a topological space such that every $\mathcal{T}$-covering mapping onto Y is hereditarily quotient mapping. Let X be the disjoint union of all subspaces of Y which belong to $\mathcal{T}$ and let $f : X \to Y$ be the natural mapping. Then X is certainly a T' space. Further, f is a $\mathcal{T}$-covering mapping which then, by hypothesis, is hereditarily quotient mapping. But then, since by I.5.2 (b) a hereditarily quotient image of a T' space is a T' space, it follows that Y is a T' space.

<u>Only if</u> : Let $f : X \to Y$ be a $\mathcal{T}$-covering mapping onto a T' space Y but yet not hereditarily quotient. Then there is a point $y \in Y$ and an open neighbourhood U of $f^{-1}(y)$ such that y does not belong to Int. $f(U)$. Then $y \in Y$ - Int. $f(U) = (Y - f(U))^{-}$. Hence there is a $\mathcal{T}$-subspace K of Y such that $y \in \{(Y - f(U) \cap K\}^{-} = (K - f(U))^{-}$. Also, there is a $\mathcal{T}$-set L in X such that $f(L) = (K - f(U))^{-}$. Since $K - f(U) \subset f(L) - f(U) \subset f(L - U)$ and $f(L - U)$ is closed, $f(L) = (K - f(U))^{-} \subset (f(L - U))^{-} = f(L - U)$. Hence y belongs to $f(L - U)$ and $f^{-1}(y) \cap (L - U) \neq \Phi$. This is a contradiction since $f^{-1}(y) \subset U$.

<u>Corollaries</u> 1) (Siwiec and Mancuso [29]) A Hausdorff space Y is a k' space iff every compact covering mapping onto Y is a hereditarily quotient mapping .

2) (Siwiec [30]) A Hausdorff space Y is Fréchet iff every sequence covering mapping onto Y is a hereditarily quotient mapping.

3) If countably compact subsets of Y are closed, then Y is quasi-k' iff every countably compact covering mapping onto Y is hereditarily quotient mapping .

III.2.2 <u>Theorem</u> A topological space Y is a T space if and only if every $\mathcal{T}$-covering mapping onto Y is a quotient mapping.

<u>Proof</u> : <u>If</u> : The proof of this part is very much similar to that of the 'if' part of III.2.1 and is hence omitted .

<u>Only if</u> : Let $f : X \to Y$ be a $\mathcal{T}$-covering mapping onto a T space Y . Let B be a subset of Y with $f^{-1}(B)$ closed in X . To prove that B is closed in Y , we must show that $B \cap C$ is closed in C for every $\mathcal{T}$-set C in Y . But $C = f(K)$ for some $\mathcal{T}$-set $K \subset X$, so that $f^{-1}(B) \cap K$ is a $\mathcal{T}$-set since $\mathcal{T}$ is closed hereditary and hence so is its image $B \cap C$. It follows that $B \cap C$ is closed in C .

<u>Corollaries</u> 1) (Siwiec and Mancuso [29]) . A Hausdorff space Y is a k space iff every compact covering mapping onto Y is a quotient mapping.

2) (Siwiec [30]) . A Hausdorff space Y is sequential iff every sequence covering mapping onto Y is a quotient mapping.

3) If countably compact subsets of a space Y are closed, then Y is quasi-k iff every countably compact covering mapping onto Y is a quotient mapping.

4) If countable subsets of a space Y are closed, then Y is a c space iff every continuous mapping onto Y is a quotient mapping. In other words, Y is a discrete topological space iff Y is a c space in which every countable set is closed. (Of course, this fact can be seen directly.)

III.2.3 <u>Theorem</u> A topological space Y is a T_R space if and only if every $\mathcal{T}$-covering mapping onto Y is a $\mathbf{T}$-weak quotient mapping.

<u>Proof</u> : <u>If</u> : The proof of this part is very much similar to that of the 'if' part of III.2.1 and hence omitted.

<u>Only if</u> : Let Y be a T_R space and let $f : X \to Y$ be a $\mathcal{T}$-covering mapping onto Y . We must show that f is a T-weak quotient mapping.

To see this, let A be a T-g-set in Y and let $f^{-1}(A)$ be closed in X . We must show that A is closed in Y . But since Y is a T_R space it suffices to prove that A is a T-G-set. That is, in fact it suffices to prove that A is T-closed.

Now consider f as a mapping from TX onto TY . Then since Y and TY as also X and TX have the same $\mathcal{T}$-subsets, it follows that f is a $\mathcal{T}$-covering mapping from TX onto TY if we can prove that $f : TX \to TY$ is continuous. Hence, let G be T-open in Y . Consider $f^{-1}(G)$. Let K be any $\mathcal{T}$-set in X . Then $f(K) \in \mathcal{T}$ whence $G \cap f(K)$ is open in $f(K)$. But then $f^{-1}(G) \cap K$ is open in K . Thus $f^{-1}(G)$ is T-open in X . Hence $f : TX \to TY$ is a $\mathcal{T}$-covering mapping. But then by III.2.2 $f : TX \to TY$ is a quotient mapping. This means that A is T-closed as $f^{-1}(A)$ is known to be closed.

<u>Corollaries</u> 1) A Hausdorff space Y is a k_R space iff every compact covering mapping onto Y is a compact-weak quotient mapping.

2) A Hausdorff space Y is an S_R space iff every sequence covering mapping onto Y is a sequence-weak quotient mapping.

3) If countably compact subsets of a space Y are closed, then Y is a quasi-k_R space iff every countably compact covering mapping onto Y is a countably compact-weak quotient mapping.

4) If countable subsets of a space Y are closed, then Y is a c_R space iff every continuous mapping onto Y is a countable-weak quotient mapping.

III.3 All discussion in previous section assumed, among other things, that all $\mathcal{T}$-subsets in the range space Y were closed. As we have already seen, such a restriction makes the underlying c space discrete. In fact, one can prove the following without any conditions.

III.3.1 Theorem The following are equivalent in any topological space Y :

a) Y is a c $(=c')$ space.

b) Every cluster covering mapping onto Y is a hereditarily quotient mapping.

c) Every cluster covering mapping onto Y is a quotient mapping.

Proof ; a) $\rightarrow$ b) : Let Y be a c space and let $f : X \rightarrow Y$ be a cluster covering mapping onto Y . Then if f is not hereditarily quotient mapping, there exists $y \in Y$ and open neighbourhood U of $f^{-1}(y)$ such that $y \notin \text{Int. } f(U)$. Then $y \in \overline{(Y - f(U))}$. Then there

is a sequence (y_n) in $Y - f(U)$ clustering to y . (Recall that $c = c'$.) But then there is a sequence (x_n) and a point x of X such that $x_n \in f^{-1}(y_n)$ for every n , $x \in f^{-1}(y)$, and $x_n \rightsquigarrow x$. Now since $x \in f^{-1}(y)$ and $f^{-1}(y) \subset U$, $x_n \rightsquigarrow x$ implies that (x_n) is frequently in U . However, $(y_n) \subset Y - f(U)$ implies that for every n , $x_n \notin U$. We thus have a contradiction.

b) $\to$ c) : Obvious.

c) $\to$ a) : Let X be the disjoint union of all clustering sequences in Y and let $f : X \to Y$ be the natural mapping. Under the present situation f turns out to be a quotient mapping from a c space whence Y becomes a c space.

III.3.2 <u>Theorem</u> A topological space Y is a c_R space if and only if every cluster covering mapping onto Y is a countable-weak quotient mapping.

<u>Proof</u> : The proof of this theorem is very much similar to that of III.2.3 and is hence omitted. Of course, while proving it, one will have to use III.3.1 rather than III.2.2.

CHAPTER IV

SUBSPACES

IV.1 Franklin [12] has observed that a Fréchet space is hereditarily Fréchet, a sequential space is closed hereditarily sequential, and a hereditarily sequential space is Fréchet. These results can be extended rather nicely to include the S_R spaces and generalized to T', T, T_R spaces. This is accomplished in the following two theorems. (Mrówka [24] has observed that (a) and (b) in Theorem IV.1.1 are equivalent.)

IV.1.1 Theorem If $\mathcal{T}$ is closed hereditary, then the following are equivalent for any T_1 space X:

a) X is a T space.

b) Every closed subspace of X is a T space.

c) Every T-closed subspace of X is a T_R space.

Proof: Clearly a) $\to$ b) $\to$ c). To show that c) $\to$ a), let $F \subset X$ be T-closed and suppose $x \in \overline{F} - F$. Then $F \cup \{x\}$ is T-closed and hence a T_R space. If there is no $\mathcal{T}$-subset K of $F \cup \{x\}$ containing x such that $x \in (F \cap K)^{-}$, then the characteristic function of $\{x\}$ is T-continuous but not continuous on $F \cup \{x\}$. This is impossible; hence F is closed.

Note that the equivalence of a) and b) does not need T_1 .

IV.1.2 <u>Theorem</u> If $\mathcal{T}$ is hereditary, the following are equivalent for any T_1 space X :

a) X is a T' space .

b) Every subspace of X is a T' space .

c) Every subspace of X is a T space .

d) Every subspace of X is a T_R space .

<u>Proof</u> : Clearly a) $\to$ b) $\to$ c) $\to$ d) . To show that d) $\to$ a) , let $F \subset X$ and let $x \in \overline{F} - F$. As in the proof of Theorem IV.1.1 , this would entail the existence of a $\mathcal{T}$-subset K of $F \cup \{x\}$ such that $x \in (K \cap F)^-$, which proves that X is a T' space .

Note that the equivalence of a) , b) and c) does not need T_1 .

The separation axiom T_1 is really needed in both IV.1.1 and IV.1.2 . To see this, let $\mathcal{T}$ consist of finite spaces so that $\mathcal{T}$ is hereditary. Let X be a countably infinite set consisting of distinct points $a , b_1 , b_2 , \ldots , b_n , \ldots$. Topologize X by calling a subset of X open iff it is empty or has the form $\{a , b_k , b_{k+1} , \ldots\}$. Then every subspace of X is T_R . But X is not a T space, since $\{b_1 , b_2 , b_3 , \ldots\}$ is a T-closed set in X which is not closed .

(Note that only constant real-valued functions on X are continuous .)

The conditions on $\mathcal{T}$ in IV.1.1 and IV.1.2 cannot be significantly weakened. For example, every compact Hausdorff space is a k' space, but not all subspaces of compact Hausdorff spaces (i.e., not all Tychonoff spaces e.g. Aren's space discussed in VI .2.4) are k' spaces, or even k_R spaces, so that IV.1.2 cannot be much improved. Likewise, to see that IV.1.1 cannot be improved, let $\mathcal{T}$ be all connected spaces. Let Y be any totally disconnected non-discrete T_2 space, and let X be the cone ΛY over Y . Then X is connected and hence a T space, but Y is a closed subspace of X and is not a T space. Indeed, with this definition of $\mathcal{T}$, a totally disconnected space will be a T space iff it is discrete.

The following results now become corollaries to theorems IV.1.1 and IV.1.2:

<u>Corollary</u> a) X is sequential iff every sequentially closed subspace of X is an S_R space.

b) X is a k space iff every k-closed subspace of X is a k_R space.

c) X is a c space iff every c-closed subspace of X is a c_R space.

d) X is a quasi-k space iff every quasi-k-closed subspace of X is a quasi-k_R space.

<u>Corollary</u> a) X is a Fréchet space iff every subspace of X is sequential iff every subspace of X is an S_R space.

b) X is a Fréchet space iff every subspace of X is a k space iff every subspace of X is a k_R space.

c) X is a c space iff every subspace of X is a c space iff every subspace of X is a c_R space.

IV.2 A closed or open subspace of a T_R space need not be T_R. In fact, one can give examples to show that in none of S_R , c_R , k_R , or quasi-k_R spaces are closed subspaces necessarily of the same kind.

IV.2.1 <u>Example</u> This example shows that a closed subspace of an S_R space need not be even c_R (thus a closed subspace of an S_R space need not be S_R and a closed subspace of a c_R space need not be c_R).

Let Ω be the ordinals $\leq \omega_1$, the first uncountable ordinal. Then Ω is a compact non-c_R space (the characteristic function of $\{\omega_1\}$ is continuous on every countable subspace of Ω but not continuous on Ω). But Ω is the Stone-Čech compactification of $\Omega_o = \Omega - \{\omega_1\}$ and, as such can be embedded as a closed subspace of $I^{C^*(\Omega_o)}$, where $C^*(\Omega_o)$ denotes the set of all real-valued bounded continuous functions on Ω_o . Since the cardinal of $C^*(\Omega_o)$ is "small" $I^{C^*(\Omega_o)}$ is an S_R space by the Mazur-Noble Theorem (I.3.2).

IV.2.2 Example This example shows that a closed subspace of a k_R space need not be even quasi-k_R (thus a closed subspace of a k_R space need not be k_R and a closed subspace of a quasi-k_R space need not be quasi-k_R).

Noble [26] has proved that a Tychonoff space can be always embedded as a closed subspace of a pseudo-compact k_R space. Since there exist countable Tychonoff spaces which are not k_R, e.g. Aren's space discussed in VI .2.4 , the claim made in the previous paragraph stands. (To see that Aren's space X is non-quasi-k_R, note first that a subset of X is compact iff countably compact iff finite, Further, the characteristic function of the set $\{(0,0)\}$ is continuous on every countably compact set but not continuous.)

These examples raise the question whether 'T-closed' in Theorem IV.1.1 can be replaced by 'closed'. This question remains unanswered.

IV.2.3 Example Let $\mathcal{T}$ denote the class of all connected spaces.

Consider the example of Knaster and Kuratowski [17] . We describe the construction briefly: Consider the Cantor set C obtained by deleting a countable collection of open intervals ('middle thirds') from the unit interval I . Let Q be the set of endpoints of these intervals (so $Q \subset C$) and set $P = C - Q$. Let $p \in R^2$ be the point $(1/2,1/2)$ and for each $x \in C$, denote by L_x the straight line segment joining p and x . Define

$$L_x^* = \{(x_1,x_2) \in L_x : x_2 \text{ is rational}\} , \text{ if } x \in Q ,$$

$$\text{and } L_x^* = \{(x_1,x_2) \in L_x : x_2 \text{ is irrational}\} , \text{ if } x \in P .$$

Then the subspace $K = \bigcup_{x \in C} L_x^*$ of R^2 is connected, while $K - \{p\}$ is totally disconnected.

It is obvious that with $\mathcal{T}$ as above, K is strongly T'. However, $K - \{p\}$ is non-discrete totally disconnected and hence cannot be T_R. (In fact, a totally disconnected space X is a T_R space where $\mathcal{T}$ stands for the class of all connected spaces iff X is discrete. To see this, consider the identity mapping of X onto itself where the domain X has the given totally disconnected topology while the range X has the discrete topology and recall that a topological space is a T_R space iff every T-continuous function defined on it with values in any arbitrary Tychonoff space is continuous.)

Thus, in general an open subspace of a strongly T' space need not be even T_R. However, we do not know whether an open subspace of an S_R, c_R, k_R, or quasi-k_R space has to be the same kind.

IV.2.4 There are non-sequential S_R spaces whose every open subspace is S_R: in fact, 2^R is an S_R space by Mazur-Noble theorem where 2 denotes the two-element discrete space, and since every basic open set in 2^R is homeomorphic to 2^R, each such basic open set is an S_R space. Hence, every open set in 2^R is locally an S_R space and hence (see I.5.1) is an S_R space. Hence the question here is: For what spaces is every T-open subspace a T_R space?

As already seen, a T_R space can have a closed or an open non-T_R subspace. However, the following theorem holds.

IV.2.5 <u>Theorem</u> If $\mathcal{T}$ is open hereditary and closed hereditary, every subspace Y of a T_R space X which is both T-open and T-closed is T_R.

<u>Proof</u> : If $F \subset Y$ is a T-G-subset of Y, it can be easily seen that F is a T-G-set in X. The result then follows by II.4.1 .

The following example given by Á. Császár shows that IV.2.5 cannot hold in general without suitable conditions on $\mathcal{T}$:

Let $\mathcal{T}$ denote the class of uncountable spaces and finite spaces. If $X = (Q \cap (0,1)) \cup (2,3)$ where Q is the set of rationals, with the usual topology, then X is T_R but $Q \cap (0,1)$ (which is both open and closed subspace of X) is not .

IV.3 Weddinton [34] has proved several facts about subspaces of k and k' spaces. We will show in this section that $\mathcal{T}$ versions of his proofs yield similar results for subspaces of T and T' spaces. However, to do so we need a condition on the class $\mathcal{T}$ of spaces under consideration and also a condition on the spaces we will consider. Hence throught this section , we will work in a setting in which

i) <u>$\mathcal{T}$ is closed hereditary</u>

<u>and</u> ii) <u>every space considered below has all its $\mathcal{T}$-subspaces</u>

closed.

IV.3.1 Definition Let X be a topological space and $A \subset X$. Then A will be said to have the property T if a subset of A is closed in A whenever it intersects every $\mathcal{T}$-subset K of X in a set closed in $A \cap K$.

IV.3.2 Proposition A subspace A of X is a T space iff

i) A has property T,

and ii) $A \cap K$ is a T space for each $\mathcal{T}$-subset K of X.

Proof : Only if : i) holds obviously. Further, A meets each $\mathcal{T}$-subset of X in a closed subset of A and hence in a closed subspace of A. But since A is a T space, the intersection of A with such a $\mathcal{T}$-subset is a T space. (ii) follows.

If : Let U be a subset of A which intersects every $\mathcal{T}$-subset of A in a closed set and let C be a $\mathcal{T}$-subset of X. We have then $A \cap C$ to be a T space. But then $U \cap C$ is closed in $A \cap C$. (For, if D be a $\mathcal{T}$-subset of $A \cap C$, D is also a $\mathcal{T}$-subset of A. But then $U \cap D$ is closed in D. Hence $U \cap C$ is closed in $A \cap C$.) Since A has property T, U is closed in A and A is a T space.

Corollary 1: Every open subspace of a T space in which $\mathcal{T}$-subsets are regular is a T space.

Proof : Let V be an open subspace of a T space X and U be a subset of V such that $U \cap K$ is open in $V \cap K$ for every $\mathcal{T}$-subset K of X. Then since V is open, $U \cap K$ is open in K. But then since X is a T space, U is open in X and hence open in V, proving that V has property T. Also, by regularity of $\mathcal{T}$-subspaces of X and closed-heredity of $\mathcal{T}$, one sees that $V \cap K$ is locally $\mathcal{T}$ and hence certainly a T space. The corollary now becomes obvious.

Corollary 2 : If X is a topological space in which every $\mathcal{T}$-subspace is regular, and if further every point of X is interior to a T subspace of X, then X is a T space.

Proof : Let A be a T-closed subset of X. Let $x \in \overline{A}$. Then since every point is interior to a T subspace of X, it follows by Corollary 1 that there is an open T subspace U of X containing x.

Now let K be a $\mathcal{T}$-subset of $U \cap \overline{A}$. Since $K \cap A$ is closed, $K \cap (A \cap U)$ is closed in $U \cap \overline{A}$.

Now $U \cap \overline{A}$ is a closed subspace of a T space U and hence a T space by the fact that if $\mathcal{T}$ is closed hereditary, a closed subspace of a T space is a T space. Hence since $K \cap (A \cap U)$ is closed in $U \cap \overline{A}$ for every $\mathcal{T}$-subset K, $A \cap U$ is closed in $U \cap \overline{A}$. Therefore, since $x \in U \cap \overline{A} \cap \overline{(A \cap U)} = A \cap U$, it follows that A is closed which was to be proved.

IV.3.3 Proposition A topological space X is a T' space if and only if every subset of X has property T.

<u>Proof</u> : <u>Only if</u> : Let A be a subset of X which is a T' space. Let U be a subset of A such that $U \cap K$ is closed in $A \cap K$ for every $\mathcal{T}$-subset K of X. If $x \in$ closure of U in A, x also belongs to the closure of U in X. Hence since X is a T' space, there exists a $\mathcal{T}$-subset K of X such that $x \in (U \cap K)^{-}$, closure being with respect to X. But $U \cap K$ is closed in A (as $U \cap K$ is a subset closed in $A \cap K$ and $A \cap K$ is closed in A – recall that K is closed in X.) Hence $x \in U \cap K$ from which it follows that $x \in U$, that is, U is closed in A. The property T is thus established.

<u>If</u> : Suppose that X is not a T' space. Then there is a subset A of X and a point $x \in \bar{A}$ such that $x \notin (A \cap K)^{-}$ for every $\mathcal{T}$-subset K of X. If K is a $\mathcal{T}$-subset of X, then $A \cap K = (A \cap K)^{-} \cap \{K \cap (A \cup \{x\})\}$. Since $A \cup \{x\}$ has property T, A is closed in $A \cup \{x\}$ which contradicts $x \in \bar{A}$.

IV.3.4 <u>Proposition</u> A subspace A of a T' space X is T' if and only if $A \cap K$ is a T' space for each $\mathcal{T}$-subset K of X.

<u>Proof</u> : <u>Only if</u> : Let A be a T' subspace of a T' space X. Let K be a $\mathcal{T}$-subset of X. Let B be a subset of $A \cap K$. If x belongs to the closure of B in $A \cap K$, x belongs to the closure of B in A. Then it follows that $A \cap K$ is a T' space.

<u>If</u> : Let $B \subset A$ and $x \in$ the closure of B in A. Then since X is a T' space, there exists a $\mathcal{T}$- subset K of X such

that $x \in (B \cap K)^-$. Then one has $x \in (B \cap K)^- \cap A \subset \overline{B} \cap (A \cap K)$. Thus there is a $\mathcal{T}$- subset C of A for which $x \in (B \cap C)^-$. (This is because $A \cap K$ is a T' space.) It follows that A is T' space.

<u>Corollary</u> 1 : If X is a T' space, then every open subspace of X is also a T' space, it being assumed that $\mathcal{T}$-subspaces are regular.

<u>Proof</u> : Let O be an open subspace of X . To prove that O is a T' space it is sufficient to prove that $O \cap K$ is a T' space for every $\mathcal{T}$-subset K of X .

Let, hence, $A \subset O \cap K$ where K is a $\mathcal{T}$-subset of X and let x be a closure point of A in $O \cap K$. We must prove that there is a $\mathcal{T}$-set K' in $O \cap K$ such that $x \in$ closure of $K' \cap A$ in $O \cap K$.

There is a set N_x open in $O \cap K$ containing x such that $\overline{N}_x \subset O \cap K$. Then $\overline{N}_x \cap K$ is a $\mathcal{T}$-set in $O \cap K$. Hence we are done if we prove that $x \in \{(\overline{N}_x \cap K) \cap A\}^-$. But since $A \subset K$, $\{(\overline{N}_x \cap K) \cap A\}^- = (\overline{N}_x \cap A)^-$. Now if G is a neighbourhood of x , $(G \cap N_x) \cap A \neq \phi$ whence $G \cap (\overline{N}_x \cap A) \neq \phi$. It follows that x belongs to $(\overline{N}_x \cap A)^- = \{(\overline{N}_x \cap K) \cap A\}^-$.

<u>Corollary 2</u> : If each point of X is interior to a T' space, X is a T' space, it being assumed that $\mathcal{T}$-subspaces are regular.

<u>Proof</u> : To prove that X is a T' space it suffices to prove that every subset A of X has the property T by IV.3.3 . Hence consider

a subset A of X. Let $B \subset A$ be such that $B \cap K$ is closed in $A \cap K$ for every $\mathcal{T}$-subset K of X. We must prove that B is closed in A.

Let x be a closure point of B in A. Then, firstly, since each point of X is interior to a T' space by hypothesis, it follows by Corollary 1 above that there is an open subspace G of X containing x which is a T' space. Then, since $x \in \overline{(G \cap B)}$, there exists a $\mathcal{T}$-set K' such that $x \in \overline{\{(G \cap B) \cap K'\}}$.

Thus, $x \in \overline{(B \cap K')}$. But then, since $B \cap K'$ is closed in $A \cap K'$ which is closed in A, it follows that $x \in B \cap K' \subset B$. Hence B is closed in A which was to be proved.

CHAPTER V

PRODUCTS

V.1 The behaviour of products of T_R spaces is very disappointing. Not only is it true that the product of two S_R , k_R or quasi-k_R spaces is not in general of the same kind, but the product of familiar countable Fréchet spaces need not be even a quasi-k_R space. The following example illustrates this fact.

V.1.1 <u>Example</u> This example is indeed due to Franklin [12] . Thanks are, however, due to Professor S. Willard for showing the author that this example possesses the properties mentioned previously.

For the sake of simplicity, we break consideration into two parts.

I Let $\mathcal{Q}$ be the rationals in $(-1,1)$, Q' the rationals in R with integers identified, and let $X = \mathcal{Q} \times Q'$. X then is the product of two Fréchet spaces.

Let now (x_n) be a sequence of irrationals < 1 converging monotonically downward to 0 . For $n = 0 , 1 , 2 , \ldots$, let T_n be interior of plane triangle determined by points (x_n,n) , $(1,n+\frac{1}{2})$, $(1,n-\frac{1}{2})$ and T_n' the reflection of T_n in the y-axis. Let R_n be interior of rhombus determined by points $(-x_n,n)$, $(0,n+\frac{1}{2})$, (x_n,n) and $(0,n-\frac{1}{2})$. Then $W_n = T_n \cup R_n \cup T_n'$ is an open subset of the

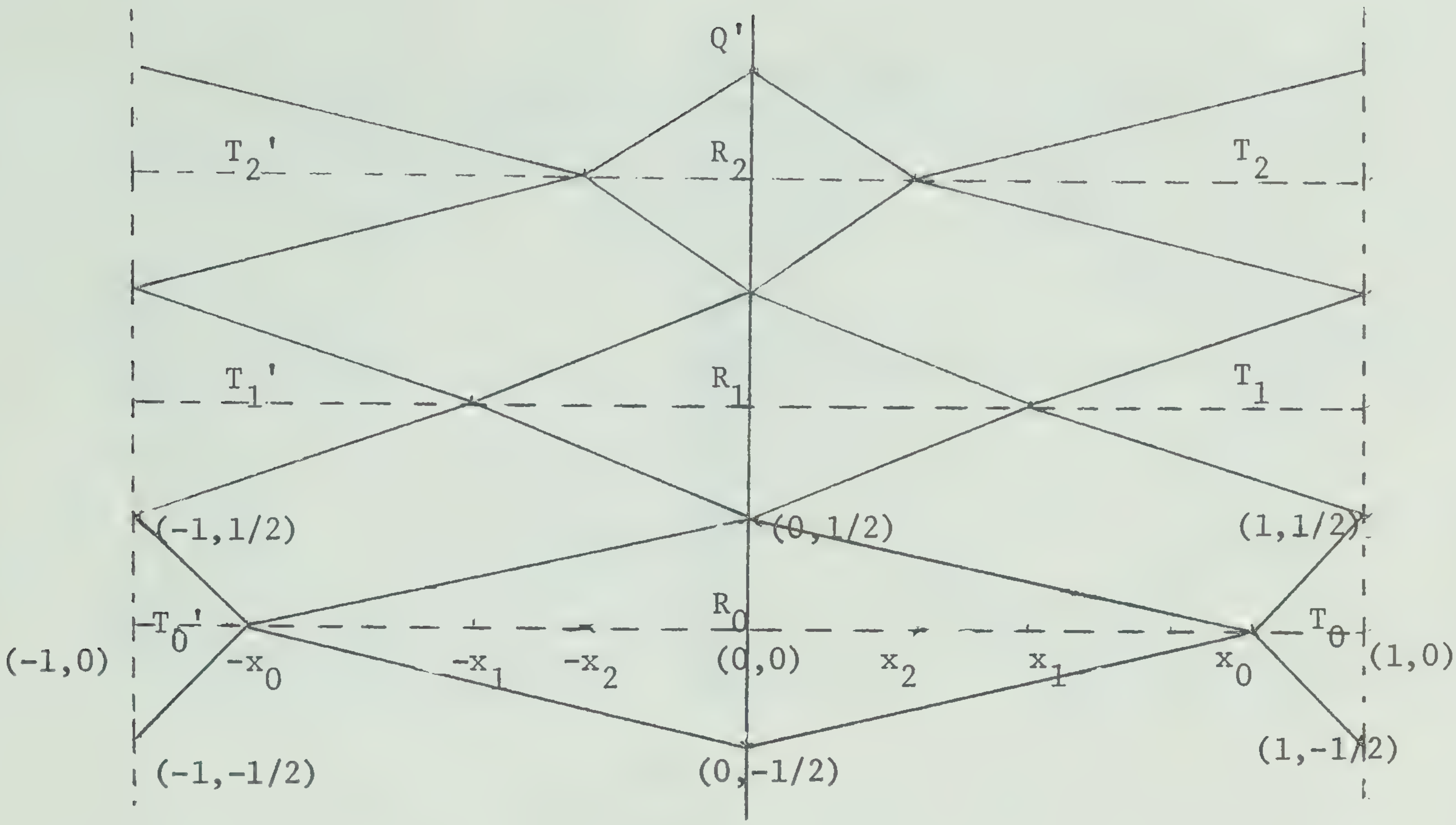

Fig. 2

plane. Thinking of X as a subset of the plane with horizontal integer lines identified, let $W = X \cap (\bigcup_{n=-\infty}^{\infty} W_n)$ where $W_{-n} = T_{-n} \cup R_{-n} \cup T'_{-n}$, T_{-n} , R_{-n} , T'_{-n} being reflections of T_n , R_n , T_n' respectively in the x-axis.

If $P_1 : X \to \mathcal{Q}$ and $P_2 : X \to Q'$ are the projections, for any neighbourhoods U and U' of 0 in $\mathcal{Q}$ and Q' respectively, $P_1^{-1}(U) \cap P_2^{-1}(U')$ cannot be contained in W . Hence $(0,0)$ is not an interior point of W which, therefore, cannot be open.

Now on each S_n define $f_n : S_n \to [0,1/2]$ as described below, S_n being a strip of X as shown in the figure which follows:

$f_n(x,y) = 1/2$, when $(x,y) \in \ell$, ℓ' or the unshaded region

and $f_n(x,y) = 0$, when $y = n$, $n-1$.

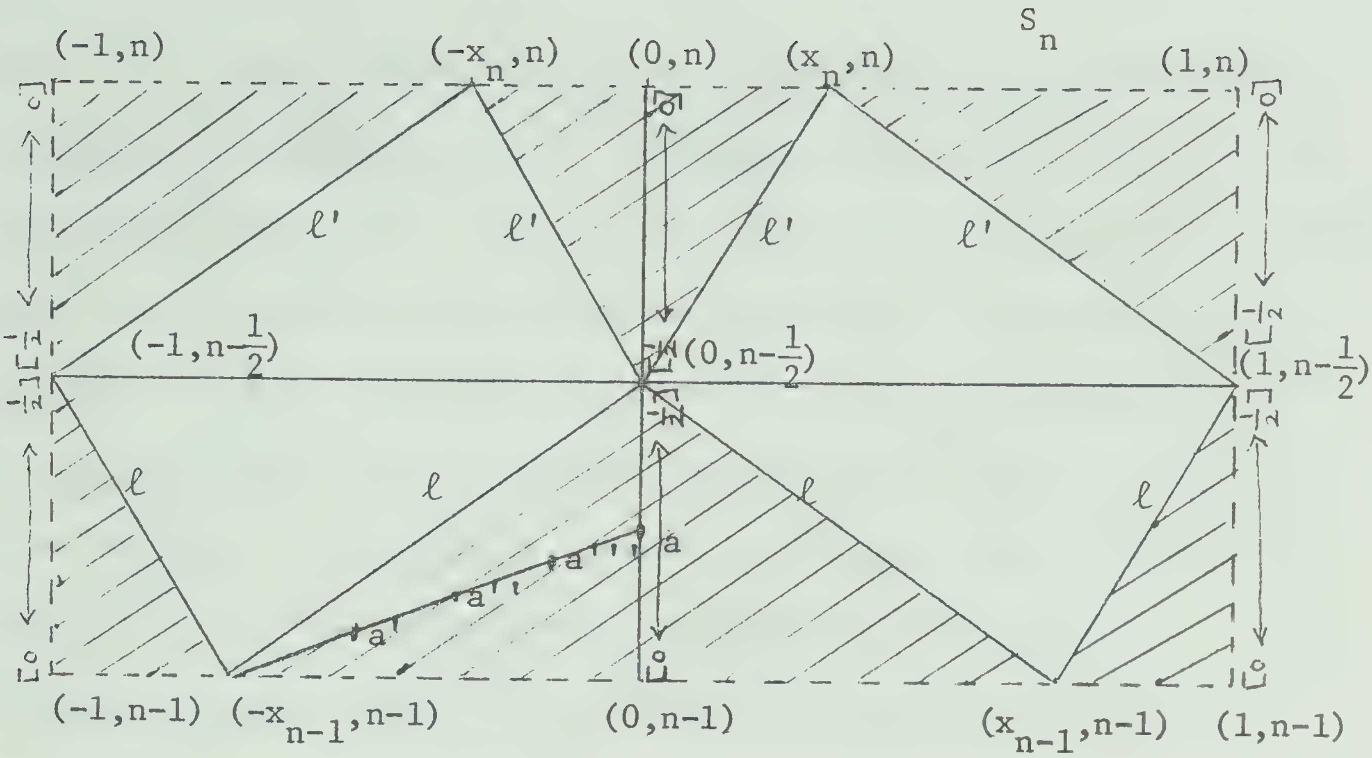

Fig. 3

At the points (x,y) not covered in the above, define f_n by means of projections from the points $(-x_{n-1}, n-1)$, $(x_{n-1}, n-1)$, $(-x_n, n)$, (x_n, n) as shown in the figure, certain vertical segments being identified with $[0,1/2]$. For example, the points a', a'', a''', etc. are mapped onto a real number in $[0,1/2]$ with which the point a is identified for the purpose of defining f_n.

It can be easily verified that f_n thus defined is continuous on S_n.

Now define $f : X \to [0,1/2]$ by setting $f \mid S_n = f_n$ for every n.

f then is not continuous, since $f^{-1}\{[0,1/2)\} = W$ which is not open.

Now to prove that $\mathcal{Q} \times Q'$ is not a quasi-k_R space we must show that f is continuous on every compact subset $K \subset \mathcal{Q} \times Q'$. (Note

that $\mathcal{Q} \times Q'$ being countable there is no distinction between compact and countably compact subsets.) For this, it is sufficient to observe that every compact set K of $\mathcal{Q} \times Q'$ will intersect $\{S_n$ - top & bottom edges$\}$ non-trivially for only finitely many n's, since $f \mid S_n = f_n$ is continuous for every n. To prove the above observation, suppose it does not hold. Then one can pick up one point from $K \cap \{S_n$ - top & bottom edges$\}$ for infinitely many n. These points form a closed subset of K and is hence compact. However, one can easily see that this set is indeed non-compact, giving the necessary contradiction.

Thus $\mathcal{Q} \times Q'$ is a product of two Fréchet spaces which is not quasi-k_R.

II We are now in $Q' \times Q'$. Here first we extend the construction in part I to the whole of the plane by taking reflection into the line $x = 1$ of the portion between the lines $x = 0$ and $x = 1$ and repeating this process till the whole right half-plane is covered. Do the same for the left half-plane. Then identify the vertical lines $x = 0, \pm 1, \pm 2, \ldots$.

Now each f_n will be defined on the infinite strip S_n^∞ by a process similar to that described in part I. Again proceeding the same way as in part I, one gets a function f which is continuous on every countably compact subset of $Q' \times Q'$ but not continuous on $Q' \times Q'$.

Thus the square of a Fréchet space need not be even quasi-k_R.

One might, however, note that it follows by I.5.2 that whenever $\mathcal{T}$ is closed under quotient maps, if $\prod_{i\in I} X_i$ is T_R, so is each X_i.

V.2 When every factor in a product is first countable the situation is much different. This is most effectively illustrated by the theorem of Mazur and Noble (Theorem I.3.2). Noble has, in fact, proved his theorem for wider class of spaces which he calls spaces of $\mathcal{C}*$ type. The relevant definitions are given in the next paragraph.

V.2.1 Let Y be any topological space. Let $\mathcal{C}(Y)$ denote the collection of compact subsets of Y which, as subsets, have countable neighbourhood bases. (For $K \subset Y$ a neighbourhood base for K is a collection of open sets U_a such that for $V \supset K$ and V open, $K \subset U_a \subset V$ for some a.) Let $\mathcal{C}*(Y) = \{K \in \mathcal{C}(Y)$: as a space, K is first countable$\}$. A space Y is said to be <u>of type $\mathcal{C}$</u> if there is a subcollection $\mathcal{C}_o(Y)$ of $\mathcal{C}(Y)$ such that for each y in Y, each C in $\mathcal{C}_o(Y)$ with y in C, and each neighbourhood U of y, there exists a C' in $\mathcal{C}_o(Y)$ with $y \in C' \subset U \cap C$. We say that Y is of <u>type $\mathcal{C}*$</u> if $\mathcal{C}_o(Y)$ can be chosen as a subcollection of $\mathcal{C}*(Y)$.

Every first countable space is of type $\mathcal{C}*$ and every space of type $\mathcal{C}*$ is of type $\mathcal{C}$.

On the pattern of sequential cardinals of Noble [27] we define T-cardinals below.

V.2.2 <u>Definition</u> A cardinal α is called a <u>T-cardinal</u> if there exists a non-zero real-valued T-continuous function $\sigma : 2^A \to R$ which maps finite sets (of A) to zero where A is any set whose cardinality is α and 2^A is the power set of A with the topology on 2^A having for its subbase $\{\{B \subset A \mathbin{/} x \in B\} \mathbin{/} x \in A\} \cup \{\{B \subset A \mathbin{/} x \notin B\} \mathbin{/} x \in A\}$.

Since every quasi-k continuous function is k-continuous and every k-continuous function is sequentially continuous, it follows that every non-sequential cardinal is non-k and that every non-k cardinal is non-quasi-k. One can similarly see that every non-sequential cardinal is non-c. However, we do not know the relation of the non-c cardinal with the other three mentioned. Also, since 2^A is totally disconnected, it follows that every infinite cardinal is a connected cardinal. In general, the question of existence of the T-cardinals could be difficult.

V.2.3 We need two more concepts. A subspace of $X = \prod_{a \in A} X_a$ is called a <u>Σ-subspace</u> if it has the form $\{x \in X : \delta(x,y) \text{ is countable}\}$ for some fixed y in X , where $\delta(x,y) = \{a : x_a \neq y_a\}$. (Such subspaces are studied in Corson [10] .) A Σ-subspace is a <u>Σ^0-subspace</u> if each $\delta(x,y)$ is finite. A function defined on a product space $X = \prod_{a \in A} X_a$ will be called <u>Σ-continuous</u> (respectively <u>Σ^0-continuous</u>) if its restriction to each Σ-subspace (respectively Σ^0-subspace) is continuous. Also, call a function <u>2-continuous</u> if it is continuous when restricted to each subspace of the form $\prod_{a \in A} Y_a$ where for each a , $1 \leq \text{card.} (Y_a) \leq 2$.

Extending the arguments of Noble we prove a very general theorem:

V.2.4 Theorem Let $\mathcal{T}$ be closed under continuous mappings. Let $X = \prod_{a\in A} X_a$ be such that every Σ^o-subspace is a T_R space where each X_a is T_1. Suppose further that for every $Y_a \subset X_a$ with $1 \le \text{card } Y_a \le 2$, $\prod_{a\in A} Y_a$ is such that every T-continuous function on $\prod_{a\in A} Y_a$ is continuous on some Σ-subspace of $\prod_{a\in A} Y_a$.

Then whenever card A is non-T, X is a T_R space.

Proof : It is enough to show that every T-continuous function $f : X \to R$ is 2-continuous. For, since every Σ^o-subspace of X is T_R, f is Σ^o-continuous. But then this fact together with 2-continuity of f would imply continuity of f by Theorem 1.1 of Noble [27], according to which a function on a product of topological spaces into a regular space which is Σ^o-continuous and 2-continuous is continuous.

Let $Y_a \subset X_a$ with $1 \le \text{card } Y_a \le 2$ for every a. Then $f : \prod_{a\in A} Y_a \to R$ is T-continuous. Hence by hypothesis f is continuous on some Σ-subspace Y of $\prod_{a\in A} Y_a$. Then by Theorem 1 of Engelking [11] (by which if X is a product of a family of T_1 spaces such that product of every finite number of them is Lindelöf and if Y is a Hausdorff space such that the diagonal of $Y \times Y$ is a G_δ-set, then a continuous function from a Σ-subspace of X into Y extends to X continuously) $f|Y$ extends to a continuous function f^* from

$\prod_{a \in A} Y_a$ into R .

Fix some $x \in Y$ and let y be any point in $\prod_{a \in A} Y_a$ and define $\sigma : 2^A \to R$ by the rule $\sigma(B) = f(xBy) - f^*(xBy)$ where xBy is a point with co-ordinates y_a for $a \in B$ and x_a otherwise.

Since f and f^* coincide on Y , σ maps finite (indeed countable) sets to zero.

Also, σ is T-continuous. (To see this, let $\rho : 2^A \to \prod_{a \in A} Y_a$ be defined by $\rho(B) = xBy$, where xBy has the same meaning as given before. Then ρ is continuous. ....(1)
{To see this, let $\Pi R_a \subset \Pi Y_a$ be a subbasic open set of ΠY_a. Then $R_a = Y_a$ for all a except some a_o. (ΠR_a stands for $\prod_{a \in A} R_a$, etc.)

There are the following possibilities:

i) $R_{a_o} = Y_{a_o}$
ii) R_{a_o} consists of a single point ($\overline{\overline{Y}}_{a_o}$ being 2).

i) Here $\rho^{-1}(\Pi R_a) = 2^A$ and hence is open in 2^A .

ii) There are two cases here: (A) $x_{a_o} \neq y_{a_o}$

(B) $x_{a_o} = y_{a_o}$.

(A) has two subcases : (A_1) $R_{a_o} = \{y_{a_o}\}$

(A_2) $R_{a_o} = \{x_{a_o}\}$.

(A_1) : Here $\rho^{-1}(\Pi R_a)$ = the class of all subsets con-

taining a_o . For, if $a_o \in B$, then $\rho(B) = xBy$ = a point z with $z_{a_o} = y_{a_o}$. Also, if z is such that $z_{a_o} = y_{a_o}$, then $z = xBy$ holds only if $a_o \in B$.

Also, the class of all subsets of A containing a certain point of A is a subbasic open set of 2^A .

(A_2) : Here $\rho^{-1}(\Pi R_a)$ = the class of all subsets of A not containing a_o . For, if $a_o \notin B$, $\rho(B) = xBy$ = a point z with $z_{a_o} = x_{a_o}$. Also, if z is such that $z_{a_o} = x_{a_o}$, then $z = xBy$ holds only if $a_o \notin B$.

Again, the class of all subsets of A not containing a certain point of A is also a subbasic open set of 2^A .

(B) also has two subcases : (B_1) $R_{a_o} = \{y_{a_o}\} = \{x_{a_o}\}$
(B_2) $R_{a_o} \neq \{y_{a_o}\} = \{x_{a_o}\}$.

(B_1) : Here $\rho^{-1}(\Pi R_a)$ = the class of all subsets containing a_o as in case (A_1) .

(B_2) : Here $\rho^{-1}(\Pi R_a) = \phi$ obviously .}

Further, $f - f^*$ is T-continuous on ΠY_a . ...(2)

Also, T is closed under continuous mappings. .....(3)

From (1) , (2) and (3) it follows that σ is T-continuous.)

But since card A is non-T , it follows that σ is identically zero. Hence $0 = \sigma(A) = f(xAy) - f^*(xAy) = f(y) - f^*(y)$. Since y was arbitrary, $f = f^*$. Hence f is continuous on $\prod_{a\in A} Y_a$, that is, f is 2-continuous which was to be proved.

Corollaries 1) (Noble [27]) The product $\prod_{a\in A} X_a$ of T_2 spaces each of type C^* is an S_R space if card A is non-sequential.

2) The product $\prod_{a\in A} X_a$ of T_1 spaces each of type C^* is a c_R space if card A is non-c.

Incidentally, Noble [27] has proved that arbitrary product of C type spaces is always k_R .

Proofs of corollaries : The corollaries will be clear from the following theorems of Noble:

1) Each Σ-subspace of a product of first countable spaces is a Fréchet space. (Noble [27] Theorem 2.1).

2) Each Σ-subspace of a product of spaces of type C^* is a sequential space. (Noble [27] Theorem 2.4).

V.3 The following results in connection with products may be of interest:

V.3.1 Proposition A T_1 space X is discrete if and only if $X \times Y$

is a quasi-k' space for every Fréchet space Y .

<u>Proof</u> : <u>Only if</u> : This is easy to see. In fact, the product of a discrete space and a Fréchet space can be easily seen to be Fréchet.

<u>If</u> : It is sufficient to prove that if X is a non-discrete T_1 space, then there is a Fréchet space Y such that $X \times Y$ is not a quasi-k' space. We will prove this below.

Let $\{x_a : a \in A\}$ be a net converging to x such that $x_a \neq x$ for any a belonging to A . Let $Y_1 = \{(a,n) : a \in A$ and $n = 1, 2, 3, \ldots\}$. Further, let $Y = Y_1 \cup \{z\}$. The topology on Y is as follows: Y_1 is discrete and the open sets containing z contain all but a finite number of elements of each set D_a where by D_a we denote the set $\{(a,n) : n = 1, 2, 3, \ldots\}$. Then Y_1 is a Fréchet space. For, if $y \in Y$ and if (y_λ) be a net such that $y_\lambda \to y$, then one can easily get a sequence (y_n) of points of (y_λ) such that $y_n \to y$. (If y is an element of Y_1 , one can trivially carry this out. If $y = z$, $\{y_\lambda : y_\lambda \in (y_\lambda)\} \cap D_a$ consists of infinitely many points at least for one a , say a' . If $D_{a'}$ intersects $\{y_\lambda : y_\lambda \in (y_\lambda)\}$ in the set, say $\{(a',n_k) : k = 1, 2, \ldots\}$ then the sequence (a',n_k) converges to z (where we have already assumed without loss of generality that $n_k \neq n_{k'}$ when $k \neq k'$) .) On the other hand (x,z) is an accumulation point of $C = \{(x_a,(a,n)) :$ $a \in A$ and $n = 1, 2, 3, \ldots\}$. (We are now in the product space $X \times Y$ where X is non-discrete T_1 space.) However, as we will see below (x,z) is not a closure point of $C \cap K$ for any countably compact

set K .

Suppose there is a countably compact set K such that (x,z) is a closure point of $C \cap K$. Then $P_Y(K)$ where P_Y is the projection onto Y is countably compact. And since each countably compact subset intersects only finitely many of the sets D_a , it follows that $P_X(C \cap K) = \{x_{a_1}, \ldots, x_{a_m}\}$, that is, $P_X(C \cap K)$ is a finite set. But then (x,z) cannot be a closure point of $C \cap K$, for, if it is, x must be a closure point of $P_X(C \cap K)$ which cannot happen since $P_X(C \cap K)$ is finite. The contradiction proves the point.

The Proposition V.3.1 is in fact only a slight improvement of a theorem of Bagley and Weddington [7] who have proved that a T_1 space X is discrete if $X \times Y$ is a k' space for every k' space. Our proof is only a very small modification of their proof.

Note the following characterization of quasi-k space; it will be needed later:

V.3.2 <u>Proposition</u> A topological space X is a quasi-k space if and only if for each subset A and $x \in \overline{A}$, there is a closed quasi-k subspace C such that $x \in \overline{(A \cap C)}$.

<u>Proof</u> : If X is a quasi-k space and if $x \in \overline{A}$, consider $\overline{A}$. Since a closed subspace of a quasi-k space is quasi-k, one has just to take $C = \overline{A}$.

Now suppose that for each subset A and $x \in \overline{A}$ there is a closed quasi-k subspace C such that $x \in (A \cap C)^{-}$. To prove that X is a quasi-k space, let R be a subset of X such that $R \cap K$ is closed in K for every countably compact subset K of X . We must prove that R is closed.

Consider a point x of $\overline{R}$. Then there is a closed quasi-k subspace C such that $x \in (R \cap C)^{-}$.

Now let K' be a countably compact subset of C . Then K' is certainly countably compact in X . Hence $R \cap K'$ is closed in K' . In other words, $(R \cap C) \cap K'$ is closed in K' for every countably compact subset K' of C . But then since C is quasi-k subspace of X , it follows that $R \cap C$ is closed in C . Further, since C itself is closed in X , $R \cap C$ is closed in X which means that x belongs to $R \cap C$ and hence to R which was to be proved.

We are now in a position to prove the following proposition.

V.3.3 <u>Proposition</u> If X is a T_1 k' space and Y is a T_1 quasi-k' space and further if $X \times Y$ has a nested neighbourhood base at each point, then $X \times Y$ is a T_1 quasi-k' space.

<u>Proof</u> : We will need the following result from Bagley and Weddington [7] :

If $X \times Y$ has a nested neighbourhood base at $(x,y) \in \overline{A} - A$

and if there are neighbourhoods U of x and V of y such that $\{x\} \times V \cap A = \phi$ and $U \times \{y\} \cap A = \phi$, then there is a net $\{(x_a,y_a) : a \in D\}$ in A which converges to (x,y) and, for each $a_o \in D$, there are neighbourhoods R of x and S of y such that $x_a \notin R$, $y_a \notin S$ for $a \prec a_o$. Further, this net $\{(x_a,y_a)\}$ has a property that every net consisting of points of the set $\{x_a\}$ (respectively $\{y_a\}$) which converges to x (respectively y) is a subnet of the net $\{x_a\}$ (respectively $\{y_a\}$) .

Consider a subset A of $X \times Y$. Let (x,y) be a point of $\overline{A} - A$.

If the neighbourhoods U and V as in the result quoted above do not exist, then for every neighbourhood U of x $(U \times \{y\})$ intersects A non-trivially or for every neighbourhood V of y $(\{x\} \times V)$ intersects A non-trivially. Suppose that for every neighbourhood U of x , $(U \times \{y\}) \cap A \neq \phi$. This means that $X \times \{y\}$ is a closed subspace of $X \times Y$ (as X and Y are T_1) and (x,y) is a closure point of $(X \times \{y\}) \cap A$. But then since $X \times \{y\}$ is a k' space, there is a compact subset K of $X \times \{y\}$ such that (x,y) belongs to the closure of $(X \times \{y\}) \cap A \cap K$ in $X \times \{y\}$, that is, $(x,y) \in (A \cap K)^{-}$. But since K is compact in $X \times Y$ also, it follows that the condition in the definition of quasi-k' spaces is satisfied for (x,y) in this case. (If we were to suppose that for every neighbourhood V of y , $(\{x\} \times V) \cap A \neq \phi$, then the same proof would work with 'compact subset K ' being replaced by 'countably compact subset K ' . This does not affect the argument.)

Now suppose that the neighbourhoods U and V as in the result of Bagley and Weddington quoted above exist. Then there is a net $\{(x_a,y_a)\}_{a\in D}$ in A converging to (x,y) and such that for each $a_o \in D$, there are neighbourhoods R of x and S of y for which $x_a \notin R$ and $y_a \notin S$ for $a \nless a_o$. Since X is a k' space, there is a compact subset K of X such that $x \in (\{x_a\} \cap K)^{-}$. Thus there is a net $\{x_d\}$ in $\{x_a\} \cap K$ which converges to x. Then by the latter part of the result of Bagley and Weddington, $\{x_d\}$ is a subnet of the net $\{x_a\}$ and hence $\{y_d\}$ also converges to y, it being a subnet of $\{y_a\}$. Since Y is a quasi-k' space, there exists a countably compact subset C of Y such that $y \in (\{y_d\} \cap C)^{-}$. Finally, we obtain a subnet of $\{(x_a,y_a)\}$ in $(K \times C) \cap A$. Thus (x,y) belongs to the closure of $(K \times C) \cap A$. Since $K \times C$ is countably compact, it follows that $X \times Y$ is a quasi-k' space.

One might note at this point that investigation of products of topological spaces coherently determined by the class of countable spaces and the class of connected spaces is, as of the date of writing this thesis, very much desired. However, it is easy to see that the product of even locally connected spaces need not be C_R, by considering $\{0,1\}^{\aleph_0}$, $\{0,1\}$ being a two-point discrete space. The reason $\{0,1\}^{\aleph_0}$ is not C_R is that $\{0,1\}^{\aleph_0}$ is totally disconnected but not discrete as every totally disconnected C_R space should be.

CHAPTER VI

MISCELLANEOUS MATTERS AND EXAMPLES

VI.1 Before presenting examples, we prove some results concerning linearly ordered topological spaces.

Let $(X,<)$ be a LOTS (= linearly ordered topological space) . Throughout this section, for every a belonging to X , La will denote the subset $\{x \in X \ / \ x \leq a\}$ and Ra will denote the subset $\{x \in X \ / \ a \leq x\}$.

VI.1.1 <u>Proposition</u> Let $(X,<)$ be a T_R LOTS . If $a \in X$ is non-isolated in La (Ra) , then there exists a $\mathcal{T}$-subset K of X in La (Ra) such that a is an accumulation point of K , it being assumed that $\mathcal{T}$ is closed hereditary .

<u>Proof</u> : Suppose that there exists an $a \in X$ which is non-isolated in La but there is no $\mathcal{T}$-subset C of X contained in La of which a is an accumulation point.

Let f be the characteristic function of $La - \{a\}$ on

X. This function is not continuous at a. However, it is continuous on every $\mathcal{T}$-subset of X. For, if C be a $\mathcal{T}$-subset contained in either $La - \{a\}$ or Ra, then f is trivially continuous on C. Further, if C be a $\mathcal{T}$-subset of X such that $(La - \{a\}) \cap C \neq \phi$ and $Ra \cap C \neq \phi$, then $A = (La - \{a\}) \cap C$ does not contain any net converging to a. (For, if A contains a net converging to a, a will be an accumulation point of a $\mathcal{T}$-subset $A \cup \{a\}$ which is contained in La. This will contradict our assumption that there is no $\mathcal{T}$-subset C of X contained in La of which a is an accumulation point.) Further, if a net of points of C converges to a point in A then it is eventually in A and it then cannot converge to a point of Ra. Also, if a net of points of C converges to a point x in $Ra \cap C$, then it is eventually in $Ra \cap C$. (This is obvious when $x \in Ra - \{a\}$. If $x = a$, the net $(x_\lambda) \subset C$ converging to a must be again eventually in $Ra \cap C$. For, if not, (x_λ) will be frequently in A which will in turn imply that a is an accumulation point of a $\mathcal{T}$-subset $A \cup \{a\}$ contained in La and this will then contradict our assumption that there is no $\mathcal{T}$-subset C of X contained in La of which a is an accumulation point.) Also, such a net certainly cannot converge to a point of A. All this consideration shows that f is continuous on $\mathcal{T}$-subsets of X.

We have thus shown that there exists a real-valued function on X which is continuous on $\mathcal{T}$-subsets of X though not continuous on X. This contradicts the fact that X is a T_R space and proves the proposition.

<u>Corollary</u> A LOTS is first countable if and only if it is c_R .

VI .2 We now present examples:

VI .2.1 <u>Example</u> The spaces 2^R , I^I , R^R constitute easier examples of S_R spaces which are not sequential. To see that these are S_R spaces we use Mazur-Noble theorem (I.3.2) while to see that they are not sequential we note that $E = \{(x_a) : x_a = 0 \text{ or } 1 \text{ and } x_a = 0$ only countably often$\}$ is a sequentially closed subset which is not closed. In fact, (o_a) where $o_a = 0$ for every a is a closure point of E which does not belong to E . (R^R is not even a k space.)

The following example due to E. Michael [23] is more instructive:

VI .2.2 <u>Example</u> Michael has given this example to exhibit a k_R space which is not k . However, as we will see in fact it serves to exhibit an S_R space which is not even quasi-k .

First, we note a definition: Let $B \subset R^2$ with usual topology $\underset{\sim}{T}_0$ say and let $x \in B$. Then a function $f : B \to R$ is called <u>separately continuous</u> at x if $f/L \cap B$ is $\underset{\sim}{T}_0$ - continuous at x where L is either the horizontal or the vertical line through x in R^2 .

In what follows in this example, whenever $x \in R^2$, x_1 and x_2 will denote the first and the second co-ordinates of x respectively. A sequence will be denoted by, say $\{x(n)\}$. Thus the i'th co-ordinate of $x(n)$ will be denoted by $x_i(n)$.

Let X be the plane R^2 and $\underset{\sim}{T}_0$ its usual topology. Let $A \subset X$ be the x-axis. Let F be the set of all functions $f : X \to R$ which are $\underset{\sim}{T}_0$-continuous on $X - A$ and separately continuous at every point of A . Let $\underset{\sim}{T}$ be the coarsest topology making every $f \in F$ continuous. Clearly, $(X,\underset{\sim}{T})$ is a Tychonoff space with $\underset{\sim}{T}_0 \subsetneq \underset{\sim}{T}$. We will prove that $(X,\underset{\sim}{T})$ is an S_R space which is not quasi-k.

First observe that on every horizontal and every vertical line X , $\underset{\sim}{T}$ agrees with $\underset{\sim}{T}_0$.

Let $f : X \to R$ be $\underset{\sim}{T}$-sequentially continuous (i.e. sequentially continuous with respect to the topology $\underset{\sim}{T}$ on X). Since $\underset{\sim}{T}_0$ and $\underset{\sim}{T}$ agree on $X - A$, it is clear that f is $\underset{\sim}{T}_0$-sequentially continuous on $X - A$ which indeed means that f is $\underset{\sim}{T}_0$-continuous on $X - A$. To see that f is $\underset{\sim}{T}_0$-separately continuous at every $x \in A$, it will surely suffice to show that if L is a horizontal or a vertical line in X , then f/L is $\underset{\sim}{T}_0$-continuous. But f/L is $\underset{\sim}{T}_0$-sequentially continuous on L (since f is $\underset{\sim}{T}$-sequentially continuous on X and $\underset{\sim}{T}_0$ and $\underset{\sim}{T}$ coincide on L). But then f/L is $\underset{\sim}{T}_0$-continuous on L . It follows that $f \in F$ and hence f is $\underset{\sim}{T}$-continuous on X . $(X,\underset{\sim}{T})$ is thus an S_R space.

We now need some lemmas:

<u>Lemma 1</u> Let $x(n) \to x$ in $(X,\underset{\sim}{T})$ with $x \in A$. Then there exists an n such that $x_1(n) = x_1$ or $x_2(n) = x_2$.

<u>Proof</u> : See Michael [23] Lemma 3.3.

Below, a subset $Y \subset X$ will be said to be $\underset{\sim}{T}_0$ countably compact when it is countably compact in $(X,\underset{\sim}{T}_0)$ and $\underset{\sim}{T}$ countably compact when it is countably compact in $(X,\underset{\sim}{T})$.

<u>Lemma 2</u> If $C \subset X$ is $\underset{\sim}{T}$ countably compact, then there exists an $\varepsilon > 0$ and a finite $A' \subset A$ such that if $y \in C$ and $0 < |y_2| < \varepsilon$, then $y_1 = x_1$ for some $x \in A'$.

<u>Proof</u> : Replace 'compact' by 'countably compact' in the proof of Lemma 3.4 of Michael [23].

<u>Lemma 3</u> There exists a $B \subset X - A$ such that

a) if $\varepsilon > 0$, then $\{x \in B : |x_2| > \varepsilon\}$ is finite

b) B intersects each vertical line at most once

and c) if $x \in A$, each $\underset{\sim}{T}_0$ neighbourhood of x intersects B.

<u>Proof</u> : See Michael [23] Lemma 3.5.

Lemma 4 The set B above is quasi-k closed in $(X,\underset{\sim}{T})$.

Proof : This follows from Lemma 2 and a) and b) of Lemma 3 .

Now if B were $\underset{\sim}{T}$-closed, $X - B$ would be $\underset{\sim}{T}$-open with the result that if $y \in A \subset X - B$, there would exist by regularity of $\underset{\sim}{T}$, a $\underset{\sim}{T}$-neighbourhood U of y such that $y \in U \subset \overline{U} \subset X - B$. Hence to contradict our assumption that B is $\underset{\sim}{T}$-closed, we will show that if $y \in A$ and if U is a $\underset{\sim}{T}$-open neighbourhood of y , then $\overline{U}$ -- the $\underset{\sim}{T}$-closure of U in X -- is also a $\underset{\sim}{T}_0$-neighbourhood in X of some $x \in A$. The contradiction will be then clear in view of property c) of the set B .

Hence we prove the following Lemma.

Lemma 5 If $y \in A$ and if U is a $\underset{\sim}{T}$-open neighbourhood of y in X , then $\overline{U}$ -- the $\underset{\sim}{T}$-closure of U in X -- is a $\underset{\sim}{T}_0$-neighbourhood in X of some $x \in A$.

(This result appears in Michael [23] slightly differently.)

Proof : Recall first that $\underset{\sim}{T}_0$ agrees with $\underset{\sim}{T}$ on each horizontal and each vertical line L so $U \cap L$ is $\underset{\sim}{T}_0$-open in L and $\overline{U} \cap L$ is $\underset{\sim}{T}_0$-closed in X .

Let $V = \{s \in R : (s,0) \in U\}$. Then V is open in R and $V \neq \phi$ since $y \in V$.

Now for each n let

$$E_n = \{s \in R \mid (s,t) \in U \text{ whenever } |t| < 1/n\} .$$

Then $\bigcup_{n=1}^{\infty} E_n = V$.

(To see this, first suppose $s \in E_n$. Then consider $(s,0)$. From the definition of E_n , it follows that $(s,0) \in U$ and hence by the definition of V , $s \in V$. Thus $\bigcup_{n=1}^{\infty} E_n \subset V$. On the other hand, if s belongs to V , then $(s,0) \in U$. Since U is $\underset{\sim}{T}$-open in X , $U \cap L$ is open in L where L is the vertical through $(s,0)$. Hence there exists an m such that $\{s\} \times (-1/m,1/m) \subset U \cap L$. This means that $s \in E_m$. It follows that $V \subset \bigcup_{n=1}^{\infty} E_n$.)

Since V is open in R , the Baire Category Theorem implies that there is an m such that $\overline{E}_m$ has an interior point s_o in R . Let $x = (s_o,0)$ and let $W = \overline{E}_m \times (-1/m,1/m)$ where $\overline{E}_m$ may be thought of as $\underset{\sim}{T}$-closure of E_m in X . Then W is a $\underset{\sim}{T}_0$-neighbourhood of x in X , and to complete the proof we will show that $W \subset \overline{U}$.

Let $|t| < 1/m$. Then $E_m \times \{t\} \subset U$. Let $L = R \times \{t\}$. Since $\overline{U} \cap L$ is $\underset{\sim}{T}_0$-closed in L , and since L is $\underset{\sim}{T}_0$-closed, $\overline{U} \cap L$ is $\underset{\sim}{T}_0$-closed in X .

Now since $E_m \times \{t\} \subset \overline{U} \cap L$, one has $\widetilde{E_m \times \{t\}} \subset \widetilde{\overline{U} \cap L}$ ($\widetilde{\ }$ denotes closure with respect to $\underset{\sim}{T}_0$) which implies that $\widetilde{E_m \times \{t\}} \subset \overline{U} \cap L$ (as $\overline{U} \cap L$ is a $\underset{\sim}{T}_0$-closed set) which in turn implies that $\widetilde{E}_m \times \{t\} \subset \overline{U} \cap L$. But since $\underset{\sim}{T}_0 \subset \underset{\sim}{T}$, $\overline{E}_m \subset \widetilde{E}_m$. It

follows that $\overline{E}_m \times \{t\} \subset \overline{U} \cap L$.

Thus if $|t| < 1/m$, $\overline{E}_m \times \{t\} \subset \overline{U} \cap L \subset \overline{U}$. It follows that $W = \overline{E}_m \times (-1/m,1/m) \subset \overline{U}$.

We have thus exhibited a quasi-k-closed subset in $(X,\underset{\sim}{T})$ which is not closed. We have hence proved that $(X,\underset{\sim}{T})$ is not a quasi-k space.

VI .2.3 <u>Example</u> This is an example of an S_R space (and hence a c_R space) which is not a c space.

Consider $I^{C^*(\Omega_o)}$. This is an S_R space by Mazur-Noble Theorem (I.3.2) . But Ω which is a closed subspace of $I^{C^*(\Omega_o)}$ is not c_R (since the characteristic function of $\{\omega_1\}$ on Ω is not continuous though continuous on every countable subset of Ω). Since every subspace of a c space is a c space (Schedler [28]) , it follows that $I^{C^*(\Omega_o)}$ is not a c space.

VI .2.4 <u>Example</u> This is an example of a c space which is not S_R .

Consider the following space of R . Arens [2] : Let X be the set of all pairs of non-negative integers with the topology described as follows: For each point (m,n) other than $(0,0)$ the set $\{(m,n)\}$ is open. A set U is a neighbourhood of $(0,0)$ iff for all except a finite number of integers m the set $\{n : (m,n) \notin U\}$ is finite. Since every countable space is a c space, it follows that X is a c space. However, since the charac-

teristic function of $\{(0,0)\}$ is sequentially continuous but not continuous it follows that X is not an S_R space. Indeed, $(0,0)$ is a non-isolated point of X to which no non-trivial sequence converges. (In fact, any countable space which is non-S_R is an example of the point.)

VI.2.5 Example This is an example of a compact space which is not S_R.

Consider the space Ω of the ordinals $\leq \omega_1$, the first uncountable ordinal. This is a compact space which is non-S_R, since the characteristic function of $\{\omega_1\}$ is sequentially continuous but not continuous.

VI.2.6 Example The following is a countable sequential space which is not quasi-k'. This example which is in reality a modification of the above quoted example of Arens is taken from Franklin [13] (his Ex. 5.1).

Let $M = (N \times N) \cup N \cup \{0\}$ with each $(m,n) \in N \times N$ an isolated point, where N denotes the set of natural numbers. For a basis of neighbourhoods at $n_o \in N$, take all sets of the form $\{n_o\} \cup \{(m,n_o) \mid m \geq m_o\}$. U will be a neighbourhood of 0 iff $0 \in U$ and U is a neighbourhood of all but finitely many $n \in N$.

Franklin has shown that M is sequential. We will verify that M is not quasi-k'.

Before proceeding further we observe that compact subsets of M are precisely countably compact subsets of M. Also, for the sake of convenience we will denote $N \times N$ by A.

Here $0 \in \overline{A}$. We will show that there exists no compact subset K of M such that $0 \in (K \cap A)^{-}$.

If a compact subset K of M be such that $K \cap A$ is finite, then obviously $0 \notin (K \cap A)^{-}$. If a compact subset K of M contains only a finite number of points on every horizontal line in A, then one can find a neighbourhood of 0 which excludes all the points of $K \cap A$ which would imply that $0 \notin (K \cap A)^{-}$. Even if a compact subset K contains an infinite number of points on only a finite number of horizontal lines in A, there would exist a neighbourhood of 0 which would exclude all the points of $K \cap A$ whence $0 \notin (K \cap A)^{-}$. Hence if there is a compact subset K of M such that $0 \in (K \cap A)^{-}$, then K must contain infinite number of points on infinite number of horizontal lines in A, say $y = n_j$ ($j = 1, 2, 3, \ldots$). But then if (f^j, n_j) is the first point on the line $y = n_j$ which belongs to K, then consider the open covering $\{(f^j, n_j)\}_{j=1,2,\ldots} \cup \{M - \bigcup_{j=1}^{\infty} (f^j, n_j)\}$. This is an open covering of K from which no finite subcovering can be obtained whence the compactness of K is contradicted. It follows that M is not quasi-k'.

VI .2.7 <u>Example</u> This is an example of a countably compact Tychonoff space which is not even k_R.

By 3.1.5 of Frolik [14] there exists a countably compact space P such that $N \subsetneq P \subsetneq \beta(N)$ with card. $P \leq 2^{\aleph_0}$, where N is the discrete space of integers. Since every infinite closed subset of $\beta(N)$ has potency $2^{2^{\aleph_0}}$, the space P contains no infinite compact set.

Choose a non-isolated point of the space P and call it x . Consider the characteristic function of the point x defined on the space P . This function is continuous on every compact subset of P but not continuous on P . This shows that P is not a k_R space.

In fact, Michael [22] has pointed out (see his Ex. 10.6) that P is not a k space.

The following examples show that there is no relation between the connected coherence topologies and the other coherence topologies mentioned in the implication diagram on page 7.

VI .2.8 <u>Example</u> Consider the Cantor ternary set C . This is a totally disconnected compact (metric) subspace of the real line. However, since it is non-discrete it cannot be a C_R space, as a totally disconnected space which is C_R must be discrete.

VI .2.9 <u>Example</u> Let X be the real line, $\mathcal{T}_1$ the usual topology on X and $\mathcal{T}_2$ the topology of countable complements on X . Let $\mathcal{T}$ be the smallest topology generated by $\mathcal{T}_1 \cup \mathcal{T}_2$. $(X,\mathcal{T})$ is connected but not even a quasi-k_R space, since whatever be a non-isolated point x in $(X,\mathcal{T})$, one cannot find a countably compact subset K of

$(X,\mathcal{T})$ of which x is an accumulation point with the result that the characteristic function of $\{x\}$ is continuous on every countably compact subspace of $(X,\mathcal{T})$ but not continuous on $(X,\mathcal{T})$. (Note that a subset of $(X,\mathcal{T})$ is countably compact iff it is finite.) This is example 63 in "Counter-examples in Topology' of L.A. Steen and J.A. Seebach, Jr.

BIBLIOGRAPHY

[1] Arens, R., A topology of spaces of transformations, Ann. of Math. 47 (1946), 480 - 495 .

[2] ________, Note on convergence in topology, Math. Mag. 23 (1950), 229 - 234 .

[3] Arhangel'skiĭ, A., Some types of factor mappings and the relations between classes of topological spaces, Dokl. Akad. Nauk SSSR 153 (1963), 743 - 746 (= Soviet Math. Doklady 4 (1963), 1726 - 1729).

[4] ________________, Factor mappings of metric spaces, Dokl. Akad. Nauk SSSR 155 (1964), 247 - 250 (= Soviet Math. Doklady 5 (1964), 368 - 371).

[5] ________________, Bicompact sets and the topology of spaces, Trudy Moskov. Mat. Obšč. 13 (1965), 3 - 55 (= Transactions of the Moscow Math. Soc. 1965, 1 - 62).

[6] ________________, A characterization of very k spaces, Czech. Math. Jour. 18 (93) (1968), 392 - 395 .

[7] Bagley, R. W. and D. D. Widdington, Products of k' spaces, Proc. Amer. Math. Soc. 22 (1969), 392 - 394 .

[8] Birkhoff, G., On the combination of topologies, Fund. Math. 26 (1936), 156 - 166 .

[9] Cohen, D. E., Spaces with weak topology, Quart. Jour. Math., Oxford Ser. (2) 5 (1954), 77 - 80 .

[10] Corson, H. H., Normality in subsets of product spaces, Amer. Jour. Math. 81 (1959), 785 - 796 .

[11] Engelking, R., On functions defined on Cartesian products, Fund. Math. 59 (1966), 221 - 231 .

[12] Franklin, S. P., Spaces in which sequences suffice, Fund. Math. 57 (1965), 107 - 116 .

[13] _______________, Spaces in which sequences suffice II, Fund. Math. 61 (1967), 51 - 65 .

[14] Frolik, Z., Generalizations of compact and Lindelöf spaces, Czech. Math. Jour. 9 (84) (1959), 172 - 217 .

[15] __________, Generalizations of the G_δ - property of complete metric spaces, Czech. Math. Jour. 10 (85) (1960), 359-370.

[16] Hájek, O., Notes on quotient maps, Comment. Math. Univ. Carolinae, 7, 3 (1966), 319 - 323 .

[17] Knaster, B. and C. Kuratowski, Sur les Ensembles Connexes, Fund. Math. 2 (1921), 206 - 255 .

[18] Mazur, S., On continuous mappings on Cartesian products, Fund. Math. 39 (1952), 229 - 238.

[19] McArthur, W., A characterization of zero-sets, Notices Amer. Math. Soc., No. 6, Vol. 17, Oct. 1970 .

[20] Michael, E., A note on k spaces and k_R spaces, Topology Conference, Arizona State Univ. 1967, 247 - 249 .

[21] __________, Biquotient maps and Cartesian products, Ann. Inst. Fourier Grenoble 18, 2 (1968), 287 - 302 .

[22] __________, A quintuple quotient quest, Gen. Top. and its Appl. 2 (1972), 91 - 138 .

[23] __________, On k spaces, k_R spaces and $k(X)$, Pacific Jour. Math. 47 (1973), 487 - 498 .

[24] Mrówka, S., R - spaces, Acta Math. Sci. Hung. 21 (3-4) (1970), 261 - 266 .

[25] Nagata, J., Quotient and bi-quotient spaces of M-spaces, Proc. Japan Acad. 45 (1969), 25 - 29 .

[26] Noble, N., Countably compact and pseudo-compact products, Czech. Math. Jour. 19 (94) (1969), 390 - 397 .

[27] _______, Continuity of functions on Cartesian products, Trans. Amer. Math. Soc. 149 (1970), 187 - 198 .

[28] Schedler,D. A., On topologies determined by clustering sequences - A generalization of sequential spaces, Doctoral Dissert., George Washington Univ. 1971 .

[29] Siwiec, F. and V. J. Mancuso, Relations among certain mappings and conditions for their equivalence, Gen. Top. and its Appl. 1 (1971), 33 - 41 .

[30] Siwiec, F., Sequence covering and countably bi-quotient mappings, Gen. Top. and its Appl. 1 (1971), 143 - 154 .

[31] Stone, A. H., Metrizability of decomposition spaces, Proc. Amer. Math. Soc. 7 (1956), 690 - 700 .

[32] Tanaka, Y., On quasi-k spaces, Proc. Japan Acad. 46 (1970), 1074 - 1079 .

[33] Venkataraman, M., Directed sets in topology, Math. Student 30 (1962), 99 - 100 .

[34] Weddington, D. D., On k spaces, Proc. Amer. Math. Soc. 22 (1969), 635 - 638 .

[35] Whyburn, G. T., Compactness of certain mappings, Amer. Jour. Math. 81 (1959), 306 - 314 .